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NOTES ON GENERAL ANALYSIS (IV)

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(Received September 30, 1954)

In preceding papers, we discussed the variation of extended $M(r)$[1] of analytic functions in complex Banach spaces. In this note, we take up first the order of entire function in complex Banach spaces and discuss it by using extended $M(r)$ in §1.

In 1937, Professor A. E. Taylor[2] pointed out that the theorems of Weierstrass and Picard were invalid generally and showed the existence of poles of infinite orders in complex Banach spaces. Here, we investigate the isolated singular point of analytic functions in §2.

Finally, in §3, the extended lemma of Schwarz[3] will be applied to various cases which will show us the convenience of treating analytic functions in abstract spaces.

§1. The order of entire functions

Let $E_0, E_1, \cdots, E_n$ be complex Banach spaces. An E_2–valued function $f(x)$ defined in a domain (which is open and connected) in E_1 is called analytic if it is strongly continuous and admits G–differential. An E_2–valued function $h_n(x)$ defined in E_1 is called a homogeneous polynomial of degree n, if it is analytic and satisfies $h_n(\alpha x)=\alpha^n h_n(x)$ for an arbitrary complex number α.

Definition 1. *An E_2–valued function $f(x)$ defined in E_1 is called an entire function if it is analytic on whole spaces.*

Definition 2. *Put* $\rho_1=\overline{\lim_{r\to\infty}}\dfrac{\log\log M(r)}{\log r}$, *where* $M(r)=\operatorname{Sup}_{||x||=r}||f(x)||$ *and* $f(x)$ *is an entire function.*

Definiton 3. *Put* $\rho_2=\operatorname{Sup}_{||x||=1}\overline{\lim_{r\to\infty}}\dfrac{\log\log M(r,x)}{\log r}$[4], *where* $M(r,x)=\operatorname{Sup}_{|\alpha|=r}||f(\alpha x)||$ *for an arbitrary point x on the set $||x||=1$ and an entire function $f(x)$.*

Theorem 1. *If a radius of bound of entire function $f(x)$ is finite, then $\rho_1=+\infty$. If a radius of bound of en entire function $f(x)$ is infinite, then*

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|}},$$

where $f(x) = \sum_{n=0}^{\infty} h_n(x)$ *and* $h_n(x)$ *is a homogeneous polynomial of degree* n.

Proof. If a radius of bound λ of en entire function $f(x)$ is finite, $M(r) = +\infty$[5] for $r > \lambda$. Then, $\dfrac{\log \log M(r)}{\log r} = +\infty$, for sufficiently larger r such that $r > \lambda$ and $r > 1$. This shows that

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} = +\infty .$$

If $\lambda = +\infty$, then $\overline{\lim_{r\to\infty}} \sqrt[n]{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|} = 0$. Let ε be an arbitrary positive number, then there exists a positive number r_0 such that

$$M(r) < e^{r^{\rho_1+\varepsilon}},$$

for $r \geqq r_0$, from the definition of ρ_1.
On the other hand, for an arbitrary point x on the set $\|x\| = 1$, we have $f(x) = \sum_{n=0}^{\infty} h_n(x)$, since $f(x)$ is analytic on whole space E_1. Then

$$\operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \leqq \operatorname{Sup}_{\|x\|=1} \frac{M(r, x)}{r^n} \leqq \frac{M(r)}{r^n} \leqq \frac{e^{r^{\rho_1+\varepsilon}}}{r^n}. \qquad \cdots\cdots\cdots\cdots (1)$$

Since $\dfrac{e^{r^{\rho_1+\varepsilon}}}{r^n}$ takes its minimum at r_1 which satisfies $r_1^{\rho_1+\varepsilon} = \dfrac{n}{\rho_1+\varepsilon}$, the inequality (1) holds for such r_1 if n is sufficiently large. Thus we have

$$\operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \leqq \frac{e^{r_1^{\rho_1+\varepsilon}}}{r_1^n} = \left(\frac{e(\rho_1+\varepsilon)}{n}\right)^{\frac{n}{\rho_1+\varepsilon}} .$$

Taking the logarithum of the parts of two sides of the inequality,

$$\rho_1 + \varepsilon \geqq \frac{\log n - \log e(\rho_1+\varepsilon)}{\dfrac{1}{n} \log \dfrac{1}{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|}} .$$

Since $\overline{\lim_{n\to\infty}} \sqrt[n]{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|} = 0$, $\rho_1 + \varepsilon \geqq \overline{\lim_{n\to\infty}} \dfrac{\log n}{\dfrac{1}{n} \log \dfrac{1}{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|}}$,

and then we have
$$\rho_1 \geqq \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|}}, \qquad \cdots\cdots\cdots\cdots (2)$$

because ε is an arbitrary positive number.

Put $\rho = \overline{\lim_{n\to\infty}} \dfrac{n \log n}{\log \dfrac{1}{\operatorname{Sup}_{||x||=1} ||h_n(x)||}}$, then $\operatorname{Sup}_{||x||=1} ||h_n(x)|| < \left(\dfrac{1}{n}\right)^{\frac{n}{\rho+\varepsilon}}$, ... (3)

for an arbitrary positive number ε and $n \geqslant n_0(\varepsilon)$.

Then we have $\overline{\lim_{n\to\infty}} \sqrt[n]{\operatorname{Sup}_{||x||=1} ||h_n(x)||} = 0$, and we see that $\sum_{n=0}^{\infty} h_n(x)$ is an entire function. Since $\left(\dfrac{1}{n}\right)^{\frac{1}{\rho+\varepsilon}} r < \dfrac{1}{2}$ for $n \geqslant n(r)$,

$$\operatorname{Sup}_{||x||=1} ||h_n(x)|| \, r^n < \frac{1}{2^n} \quad \text{for} \quad n \geqslant \max. (n(r), n_0(\varepsilon)) .$$

Then,

$$M(r) \leqslant \sum_{n=0}^{\infty} \operatorname{Sup}_{||x||=1} ||h_n(x)|| \, r^n = \sum_{n=0}^{n(r)-1} \operatorname{Sup}_{||x||=1} ||h_n(x)|| \, r^n + \sum_{n(r)}^{\infty} \operatorname{Sup}_{||x||=1} ||h_n(x)|| \, r^n .$$

Put $c(r) = \operatorname{Sup}_{n\geqslant 0} (\operatorname{Sup}_{||x||=1} ||h_n(x)|| \, r^n)$, then $M(r) \leqslant n(r)\, c(r) + \dfrac{1}{2^{n(r)-1}}$.

Since $c(r) \leqslant e^{\frac{r^{\rho+\varepsilon}}{(\rho+\varepsilon)e}}$ (which is the maximum of $\left(\dfrac{1}{n}\right)^{\frac{n}{\rho+\varepsilon}} r^n$ for $n \geqslant 0$ and a sufficiently large r) from (3) and $n(r) \leqslant (2r)^{\rho+\varepsilon}$ from $\left(\dfrac{1}{n}\right)^{\frac{1}{\rho+\varepsilon}} r < \dfrac{1}{2}$,

$$M(r) \leqq (2r)^{\rho+\varepsilon} \, e^{\frac{r^{\rho+\varepsilon}}{(\rho+\varepsilon)e}} + \frac{1}{2} .$$

Then we have
$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} \leqslant \rho + \varepsilon ,$$

for an arbitrary positive number ε. From (2) and (4), $\rho = \overline{\lim_{r\to\infty}} \dfrac{\log \log M(r)}{\log r}$.
This completes the proof.

Theorem 2. $$\rho_2 = \operatorname{Sup}_{||x||=1} \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{||h_n(x)||}} .$$

Proof. Since $f(x)$ is an entire function, we have

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r, x)}{\log r} = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{||h_n(x)||}} ,$$

as well as $M(r)$, for an arbitrary point x on the set $||x|| = 1$. Then we have

$$\rho_2 = \operatorname{Sup}_{||x||=1} \overline{\lim_{r\to\infty}} \frac{\log \log M(r, x)}{\log r} = \operatorname{Sup}_{||x||=1} \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{||h_n(x)||}} .$$

Theorem 3. $\rho_2 \leqslant \rho_1$.

Proof. Since $M(r, x) \leqslant M(r)$, $\dfrac{\log \log M(r, x)}{\log r} \leqslant \dfrac{\log \log M(r)}{\log r}$.
and we have $\rho_2 \leqslant \rho_1$.

§ 2. Singular points of analytic functions.

A point x is called a singular point of $f(x)$, when $f(x)$ is not analytic in any neighbourhood of x. A singular point x is called an isolated singular point, if $f(x)$ is analytic in a neighbourhood of x droping itself. In this chapter, we research the state of an isolated singular point.

Definition 4. *Let an E_2-valued function $R_n(x)$ be analytic in $0<\|x\|<\infty$ in E_1. If $R_n(\alpha x)=\frac{1}{\alpha^n}R_n(x)$ for any complex number α, then $R_n(x)$ is called a homogeneous rational function of degree n.*

Theorem 4. *If $f(x)$ is analytic in $0<\|x\|<R$, then*

$$f(x)=\sum_{n=0}^{\infty} h_n(x)+\sum_{n=1}^{\infty} R_n(x),$$

where $h_n(x)$ is a homogeneous polynomial of degree n and $R_n(x)$ is a homogeneous rational function of degree n.

Proof. Let x be an arbitrary point in $0<\|x\|<R$ and α be a complex number. Then $f(\alpha x)$ is an analytic function of α in $0<|\alpha|<\frac{R}{\|x\|}$ and we have

$$f(x)=\frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha-1}d\alpha-\frac{1}{2\pi i}\int_{C'}\frac{f(\alpha x)}{\alpha-1}d\alpha,$$

where C is a circle such that $|\alpha|=r(>1)$ and C' is a circle $|\alpha|=r'(<1)$. Since the series $\sum_{n=0}^{\infty}\frac{f(x\alpha)}{\alpha^{n+1}}$ and $\sum_{n=1}^{\infty} f(\alpha x)\alpha^n$ converge uniformly respectively on C and C', we have

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha\right)+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C'} f(\alpha x)\alpha^n d\alpha\right)$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha\right)+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C} f(\alpha x)\alpha^n d\alpha\right),$$

because $f(\alpha x)\alpha^n$ is analytic as to α for $0<|\alpha|<\frac{R}{\|x\|}$.
Put $P_n(x)=\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha$ and $R_n(x)=\frac{1}{2\pi i}\int_C f(\alpha x)\alpha^{n-1}d\alpha$, then $P_n(x)$ is as usual a homogeneous polynomial of degree n by the uniformity of the integral and the theorem of Zorn[6]. From the analyticity of $f(\alpha x)$ in $0<\|x\|<R$ and the uniformity of convergence of the integral $\frac{1}{2\pi i}\int_C f(\alpha x)\alpha^n d\alpha$ we know that $R_n(x)$ is analytic in $0<\|x\|<\infty$, appealing to also the analytic continuation. Let ξ be an arbitrary complex number and x be an arbitrary point in E_1. Then we can take as C a circle with radius r which satisfies $0<r|\xi|\cdot\|x\|<R$. Then

$$R(\xi x)=\frac{1}{2\pi i}\int_{|\alpha|=r} f(\alpha\xi x)\alpha^{n-1}d\alpha .$$

Put $\xi\alpha=\beta$, then

$$\begin{aligned}R(\xi x)&=\frac{1}{2\pi i}\int_{C'} f(\beta x)\frac{\beta^{n-1}}{\xi^{n-1}}\cdot\frac{1}{\xi}d\beta\\&=\frac{1}{\xi^n}\cdot\frac{1}{2\pi i}\int_{C} f(\beta x)\beta^{n-1}d\beta\\&=\frac{1}{\xi^{n}}R(x),\end{aligned}$$

where C' is a circle with radius r' which satisfies $r'=r|\xi|$. Thus we see that $R_n(x)$ is a homogeneous rational function of degree n.

Now, we must research that which space has an isolated singular point. A set of points $\{x_1+\alpha y_1,\ x_2+\beta y_2)\}$, where points (x_1, x_2), (y_1, y_2) are fixed and α, β are arbitrary complex numbers, is called a 2–dimensional plane. If the intersection of a set Γ and an arbitrary 2–dimensional plane is connected or null set, Γ is called 2–dimensionally connected.

Lemma[7]. *Let $f(x_1, x_2)$ in $E_1\times E_2$ to E_3 be analytic on the boundary Γ of a bounded domain Δ of $E_1\times E_2$, where Γ is 2–dimensionally connected. Then $f(x_1, x_2)$ is analytic in Δ.*

Theorem 5. *If $f(x)$ has an isolated singular point, then E_0 is the one dimensional space with respect to complex numbers.*

Proof. By the axiom of Zermelo, complex Banach space is considered as a well ordered set. Then we can find a set S of elements which are linearly independent by the transfinite induction. If S does not consist of only an element, S is divided an element and others which span a subspace E_1 and E_2 separately. Then we have $E_0=E_1+E_2$ as a direct sum of E_1 and E_2. If we assume that 0 is an isolated singular point of $f(x)$ to simplify the notation, $f(x)$ is analytic on $\|x\|=\rho$, for sufficiently small positive number ρ. Appealing to Lemma[*], $f(x)$ is analytic in $\|x\|=\rho$ which contradicts to that 0 is a singular point of $f(x)$. Then we see that S consists of an element which shoes us E_0 is an one-dimensional space. This completes the proof.

§ 3. The application of the extended lemma of Schwarz.

In this chapter, we show that some of theorems[8] in the book "*Several comlex variables by S. Bochner and W. Martin*" are included in the extended lemma of Schwarz. In preceding papers, the lemma of Schwarz and the

Hadamard's three spheres theorem were extended to complex Banach spaces as follows:

The extended lemma of Schwarz[3]. *Let an E_2-valued function $f(x)$ defined in the sphere $||x||<R$ of E_1 be analytic and satisfy $f(0)=0$ and $||f(x)||\leqq M$ in the sphere $||x||<R$. Then*

$$||f(x)||\leqq\frac{M}{R}||x||.$$

The extended Hadamard's three spheres theorem[1]. *If $0<r_1<r_2<r_3$, $M(r_2)\leqq M(r_1)^\theta M(r_3)^{1-\theta}$, where $M(r)=\operatorname{Sup}_{||x||=r}||f(x)||$ and $\theta=\dfrac{\log r_3-\log r_2}{\log r_2-\log r_1}$ and then $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$.*

Theorem 6. *Let E_i-valued function $f_i(x)$ be analytic on $||x||\leqq 1$ in E_0 and satisfy $f_i(0)=0$, for $1\leqq i\leqq n$. Then*

$$(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}\leqq ||x||\operatorname{Sup}_{||x||=1}(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}.$$

Proof. In the product space $E_1\times E_2\times\cdots\times E_n$, the norm of $y=(y_1, y_2, \cdots, y_n)$ is defined as follows $||y||=(\sum_{i=1}^n ||y_i||^p)^{\frac{1}{p}}$, where y_i belongs to E_i, then this product space is the complex Banach spaces. Put $F(x)=(f_1(x), f_2(x), \cdots, f_n(x))$, then $F(x)$ is an $E_1\times E_2\times\cdots\times E_n$ valued function and analytic in $||x||\leqq 1$. Appealing to the extended lemma of Schwarz, we have $||F(x)||\leqq||x||\operatorname{Sup}_{||x||=1}||F(x)||$, when $\operatorname{Sup}_{||x||=1}||F(x)||<\infty$. On the other hand, $||F(x)||=(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}$.
Thus we have $(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}\leqq||x||\operatorname{Sup}_{||x||=1}(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}$.
Even if $\operatorname{Sup}_{||x||=1}||F(x)||=\infty$, our inequality is also held clearly.

Corollary. *If complex valued functions $f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha)$ are regular in $|\alpha|\leqq 1$ and satisfy $f_i(0)=0$ for $1\leqq i\leqq n$, then we have*

$$(\sum_{i=1}^n |f_i(\alpha)|^p)^{\frac{1}{p}}\leqq|\alpha|\cdot\operatorname{Max.}_{|\alpha|=1}(\sum_{i=1}^n|f_i(\alpha)|^p)^{\frac{1}{p}}.$$

Proof. Let $||\alpha||=|\alpha|$ and E_i be a complex plane, then we have this Corollary, appealing to Theorem 6.

Theorem 7. *Let E_i-valued function $f_i(x)$ be analytic on $||x||\leqq 1$ in E_0 for $1\leqq i\leqq n$, then we have*

$$\operatorname{Sup}_{||x||=r_2}(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}\leqq\{\operatorname{Sup}_{||x||=r_1}(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}\}^\theta\{\operatorname{Sup}_{||x||=r_3}(\sum_{i=1}^n ||f_i(x)||^p)^{\frac{1}{p}}\}^{1-\theta},$$

when $0<r_1<r_2<r_3\leqq 1$, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$ and $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$.

Proof. Put $F(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ and $\|F(x)\|=(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}$, then $F(x)$ is an analytic function defined on $\|x\| \leqq 1$ in E_0 and takes its values in the product space $E_1 \times E_2 \times \cdots\cdots \times E_n$.

Appealing to the extended Adamard's three spheres theorem, we have

$$\operatorname{Sup}_{\|x\|=r_2} \|F(x)\| \leqslant (\operatorname{Sup}_{\|x\|=r_1} \|F(x)\|^{\theta} (\operatorname{Sup}_{\|x\|=r_3} \|F(x)\|)^{1-\theta},$$

where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$. Since $\|F(x)\|=(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}$, we have

$$\operatorname{Sup}_{\|x\|=r_2} (\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}} \leqq \{\operatorname{Sup}_{\|x\|=r_1} (\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}\}^{\theta} \{\operatorname{Sup}_{\|x\|=r_3} (\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}\}^{1-\theta}.$$

We can easily have following Corollary.

Corollary. If complex valued functions $f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha)$ are regular in $|\alpha| \leqq 1$, then we have

$$\operatorname{Max.}_{|\alpha|=r_2} (\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}} \leqslant \{\operatorname{Max.}_{|\alpha|=r_1} (\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}}\}^{\theta} \{\operatorname{Max.}_{|\alpha|=r_3} (\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}}\}^{1-\theta},$$

when $0<r_1<r_2<r_3\leqslant 1$, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$.

In these theorems, we see that it is convenient to treat analytic functions in abstract spaces.

References.

1) I. Shimoda: Notes on general analysis III, Jour. of Gakugei, Tokushima Univ. Vol. IV, 1954.

2) A. E. Taylor: Analytic functions in general analysis, Ann. R. Scuola Norm. Sup. Pisa. (2) 6 (1937).

3) I. Shimoda: Notes on general analysis (II), Jour. of Gakugei, Tokushima Univ. Vol. III, 1953.

4) The order of vector valued entire functions defined in complex plane was studied by Prof. E. Hille. See E. Hille: Functional Analysis and Semi-groups.

5) See 1), Theorem 10 at page 8.

6) Max. A. Zorn: Characterization of analytic functions in Banach spaces, Annals of Math. Vol. 46 (1945). Theorem: Let $p(x)$ satisfy the following conditions, 1) it is G-differentiable on E, 2) for $|\xi|=1$, $\|p(\xi x)\|=\|p(x)\|$, 3) there exists an x_0 in E and real number M, σ, with $\sigma>0$, such that for $\|x-x_0\|\leq\sigma$ we have $\|p(x)\|\leq M$. Then $\|p(x)\|\leq M$ for $\|x\|\leq\sigma$.

7) See I. Shimoda and K. Iseki: General Analysis in abstract spaces. Jour. of the Osaka Institute of Soc. and Tec. Vol. 1, No. 1, 1949. In these papers, the generalization of Hartogs's theorem was described roughly as Prof. A. E. Taylor pointed out.

*) Of course, E_0 is not necessarily a prodtct space of E_1 and E_2, but the proof of Lemma is applicable for this case.

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ON A MONOID WHOSE SUBMONOIDS FORM A CHAIN[0)]

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§ 1. Introduction.

Generally the set $\mathfrak{S}$ of submonoids of a monoid[1)] M constitutes a complete lattice.[2)] Although it is of course that the structure of $\mathfrak{S}$ is given by that of M, some property of M is characterized by a property of $\mathfrak{S}$. This paper is concerned with the determination of all types of a monoid whose all submonoids form a chain. We shall call such a monoid a Γ-monoid. In case when M is a finite group, the problem is solved by R. Baer [1] *i.e.*,

Lemma 1. *The lattice formed by all submonoids of a finite group G is a chain if and only if G is a cyclic group of prime power order.*

In the present paper, it will be concluded that if a Γ-monoid M is finite, M is a certain power monoid of order n, where $p^m \leq n \leq p^m+2$, and p is a prime number, and if M is infinite, M is a limit group of finite cyclic groups of prime power order.

§ 2. Preliminaries.

In the below Lemmas 2 and 3 we assume M to be a monoid. Let us denote by $[a]$ a submonoid of M generated by only an element $a \in M$, *i.e.*,

$$[a] = \{a^i;\ i = 1, 2, 3, \cdots\}.$$

If $[a]$ is infinite (finite), then the element a is said to be an element of infinite (finite) order or an infinite (finite) element. We define a quasi-ordering $a \leq b$ as $[a] \subset [b]$.

Lemma 2. *$a \leq b$ if and only if $a = b^n$ for some positive integer n.*

Proof. If $a = b^n$ for some n, then $a^m = (b^n)^m = b^{nm} \in [b]$ for every m. Therefore $[a] \subset [b]$. The converse is clear by the definition.

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0) This research was sponsored, in part, by MIKI-KORAKUKAI.

1) The "monoid" and "submonoid" are synonyms of the "semigroup" and "subsemigroup" respectively. cf. N. Bourbaki; Structure algebriques.

2) We shall consider even the empty set as a submonoid.

Let $\bar{M}$ be a quotient set got by introducing into M the equivalence relation $a \sim b$ defined as $a \leqq b$ and $b \leqq a$. $\bar{M}$ is a partly ordered set.

Lemma 3. *There is an element b different from a such that $a \sim b$ if and only if $[a]$ is a finite cyclic group of order $n \geqq 3$.*

Proof. If $a \sim b$ as well as $a \neq b$, then $a = b^k$ and $b = a^m (k \neq 1,\ m \neq 1)$ by Lemma 2, and we have $a = a^{km}$ where $km \geqq 4$. It follows that a is of finite order and it belongs to the greatest group G of $[a]$ (see [2]). Hence we get $[a] = G$. Next, supposing that $a = a^t$, $t = 2$ or 3, it is readily led that $a = a^2 = b$. Therefore the order of G is at least 3. Conversely if $[a]$ is a cyclic group of order $n \geqq 3$, there is a positive integer m such that $1 < m < n$ and m is relatively prime to n. Then $a^m \neq a$ and $a \sim a^m$. Thus the proof of the lemma has been completed.

Hereafter we assume $\mathfrak{S}$ to be a chain, in other words, M to be a Γ-monoid, and $\mathfrak{S}$ is represented as

$$\mathfrak{S} = \{S_\gamma;\ \gamma \in \Lambda\}$$

where the set Λ of suffixes is a chain, and has 0 as the least element and ξ as the greatest, *i.e.*, $S_0 = \phi$, $S_\xi = M$, and $S_\gamma \subset S_\zeta$ for $\gamma < \zeta$.

Lemma 4. *Every submonoid of a Γ-monoid is a Γ-monoid.*

Proof. Let S be a submonoid of M and $\mathfrak{T}$ be the set of all submonoids of S. Of course $\mathfrak{T} \subset \mathfrak{S}$. The ordering in $\mathfrak{S}$ is preserved in $\mathfrak{T}$.

Lemma 5. *The homomorphic image $M' = f(M)$ of a Γ-monoid M by the homomorphism f is a Γ-monoid.*

Proof. Let S_γ' and S_ζ' be submonoids of M', and let S_γ and S_ζ their inverse images by f respectively. By the assumption, either $S_\gamma \subset S_\zeta$ or $S_\zeta \subset S_\gamma$; and so evidently $f(S_\gamma) \subset f(S_\zeta)$ or $f(S_\zeta) \subset f(S_\gamma)$. Thus M' is proved to be a Γ-monoid.

Lemma 6. *If M is a Γ-monoid, every element of M is of finite order. Namely $[a]$ is a finite power monoid.*

Proof. Suppose that there is an infinite element $a \in M$. By Lemma 4, $[a]$ is a Γ-monoid. But we see that $[a]$ has two incomparable submonoids

$$[a^2] = \{a^{2i};\ i = 1, 2, 3, \cdots\}, \quad [a^3] = \{a^{3i};\ i = 1, 2, 3, \cdots\};$$

this is contradictory with the assumption. Hence every element is of finite order.

§ 3. Type of chain.

We denote by $\mathfrak{S}'$ the set of all power submonoids of M. $\mathfrak{S}'$ is a subchain of $\mathfrak{S}$ admitting a chain Γ'' as an index set, and

$$\mathfrak{S}' = \{S_\gamma;\ \gamma \in \Gamma''\}, \quad \Gamma'' \subset \Gamma$$

where $S_\gamma \subset S_\zeta$ for $\gamma < \zeta$.
We easily have

Lemma 7. *Every S_γ is finite,*

Lemma 8. *M' is order-isomorphic with Γ''.*

The following lemma is remarkable.

Lemma 9. *The ordinal number of Γ'' is not greater than the first infinite ordinal number ω.*

Proof. When Γ'' is finite, it is evident that Γ'' has finite ordinal number. We shall discuss as to the case that Γ'' is infinite. Let σ be any element of any subset Σ' of Γ''. By Lemmas 7 and 8, the number of the elements of Σ' which lie before σ is finite; and so Σ' has a least element. In other words, Γ'' is a well-ordered set, the ordinal number of which we denote by γ. Since Γ'' is infinite, $\omega \leqq \gamma$. Next, suppose that $\omega+1 \leqq \gamma$, then it follows that S_ω is infinite. This contradicts with Lemma 7. Henceforth we have $\gamma = \omega$.

According to the above lemmas, all elements of $\mathfrak{S}'$ may be generally denoted as follows:

if M is finite, $\phi = S_{-1} \subsetneqq S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_\eta$,
if M is infinite, $\phi = S_{-1} \subsetneqq S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_\gamma \subsetneqq S_{\gamma+1} \subsetneqq \cdots$

where $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$ for no $S_\delta \in \mathfrak{S}'$ $(\gamma = -1, 0, 1, 2, \cdots)$.

An increasing sequence $\{S_\gamma\}$ of power submonoids of M where there is no power submonoid S_δ such that $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$ is called a full chain of power submonoids of M.

Lemma 10. *An increasing sequence $\{S_\gamma\}$ is a full chain of power submonoids if and only if any element of $S_{\gamma+1} - S_\gamma$ generates $S_{\gamma+1}$.*

Proof. Suppose $\{S_\gamma\}$ is a full chain of M. Set $S_\gamma = [a]$ and $T = S_{\gamma+1} - S_\gamma$. Obviously $[x] \subset S_{\gamma+1}$ for any $x \in T$, and we get $[a] \subsetneqq [x] \subset S_{\gamma+1}$. Hence $[x] = S_{\gamma+1}$. Conversely if any element of $S_{\gamma+1} - S_\gamma$ generates $S_{\gamma+1}$, it is seen that there is no S_δ such that $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$.

From Lemma 10 we obtain easily the following

Lemma 11. *If M is a Γ-monoid, then there exists a full chain*

$$[a_0] \subset [a_1] \subset \cdots \subset [a_\gamma] \subset \cdots$$

of at most countable power submonoids such that $M = \bigcup_{\gamma=0}^{\infty} [a_\gamma]$.

A full chain $\{[a_\gamma]\}$ satisfying $M = \bigcup_{\gamma=0}^{\infty} [a_\gamma]$ is called a basic chain of M. The below lemma is worthy of notice.

Lemma 12. *If A monoid M has a basic chain $\{[a_\gamma]\}$, any proper submonoid of M is a power monoid.*

Proof. Let S be any proper submonoid of M. There exists greatest $\bar{\gamma}$ of γ such that $[a_\gamma] \subset S$. For, if not so, $[a_\gamma] \subset S$ for every γ, and so $M = S$. Now, since $[c] \subset S$ for every $c \in S$, we have $c \in [c] \subset [a_{\bar{\gamma}}]$; and $S \subset [a_{\bar{\gamma}}]$. Combining it with $[a_{\bar{\gamma}}] \subset S$, we get $S = [a_{\bar{\gamma}}]$.

Thus it is concluded that every submonoid of a Γ-monoid M is no other than a power monoid which forms a full chain of M.

The following theorems are immediately obtained.

Theorem 1. *If M is a Γ-monoid, the ordinal number of $\mathfrak{S}$ is not greater than $\omega+1$, and every proper submonoid of M is a finite power monoid.*

Theorem 2. *A monoid M is a Γ-monoid if and only if M has a basic chain.*

As special case we have

Lemma 13. *If M is a Γ-monoid as well as a group, then $[a]$ is a prime power cyclic group for every $a \in M$. Moreover the order of $[a]$ is a power of the same prime number.*

Proof. Let a be any element different from the unit e of M. Of course $[a]$ is finite. We let n be the order of a:

$$a^n = e \qquad (n > 1).$$

For every $m \geqq n$, a^m belongs to the cyclic group, the greatest group G_0 of $[a]$ (see [2]).

From
$$a^n = a^{2n} = e,$$
$$aa^{n-1} = a^{n+1}a^{n-1}.$$

Since M is a group, we get $a = a^{n+1}$ by multiplying the both sides by the inverse of a^{n-1}. Hence $a \in G_0$, that is to say, $[a]$ is a cyclic group. It is owing to Lemma 1 that $[a]$ is a prime power group. The latter half of the lemma is readily shown.

§ 4. Type of difference monoid.

Lemma 14. *A Γ-monoid is unipotent inversible* [3].

Proof. If there exist distinct idempotents a and b in M, then $\{a\}$ and $\{b\}$ are incomparable submonoids of M. This conflicts with the assumption. Therefore M is unipotent. By Lemma 6, any element a is represented as $a^n = aa^{n-1} = e$ for some $n > 1$; that is, M is inversible.

According to [2] [3], $G = Me$ is the greatest group of M. We denote by M^* the difference monoid [4] of M modulo G. Then M^* is a Γ-zero-monoid [2] and every element of M^* is of finite order by Lemmas 5 and 6.

Lemma 15. *Let Z be a Γ-zero-monoid. Every element of Z is of order*[3] *at most* 3.

Proof. If there is an element x of order 4 in Z,

$$[x] = \{x, x^2, x^3, 0\}, \quad x^4 = 0,$$

contains two incomparable submonoids

$$A = \{0, x^2\} \quad \text{and} \quad B = \{0, x^3\},$$

contradicting with the definition of a Γ-monoid. If there is an element $x \in Z$ is of order $n > 4$, then a power zero-monoid $[x]$ is homomorphic onto a power zero-monoid $C = \{X, X^2, X^3, X^4 = 0\}$ [2] and the submonoids S_1 and S_2 which correspond to $\{0, X^2\}$ and $\{0, X^3\}$ respectively are incomparable.

Theorem 3. *A zero-monoid Z is a Γ-monoid if and only if Z is a power zero-monoid of order*[4] *at most* 3.

Proof. Suppose that Z is a Γ-zero-monoid. If the number of elements of a zero-monoid Z is no less than 4 or infinite, Lemma 15 makes it possible for us to find different elements x and y having equal order m where m is 2 or 3. Then it is seen that $[x]$ and $[y]$ are incomparable submonoids of Z. Hence Z is composed of at most 3 elements. Conversely we shall prove that a zero-monoid of order at most 3 is a Γ-monoid.

---

3) By the order n of an element x of a zero-monoid, we mean such n that $x^n=0$ and $x^m \neq 0$ for $1 \leqq m < n$.

4) We mean the order of a monoid M the number of elements of M.

Since a zero–monoid of order 2 is nothing but

	0	a
0	0	0
a	0	0

,

the proof of this case is trivial. Using the theory of a finite zero–monoid [2] [5], it is proved that zero–monoids of order 3 have two types as the following:

	0	a	b
0	0	0	0
a	0	0	0
b	0	0	0

,

	0	a	b
0	0	0	0
a	0	0	0
b	0	0	a

.

The former is neither a power monoid nor a Γ–monoid for $\mathfrak{S}$ is

$\{0, a, b\}$

$\{0, a\}$ $\{0, b\}$.

$\{0\}$

The latter is not only a power-monoid but a Γ–monoid. In fact, $\mathfrak{S}$ is

$\{0, a, b\}$

$\{0, a\}$

$\{0\}$.

Thus we have completed the proof.

By Theorem 3, the difference monoid M^* of M modulo G has been verified to consist of at most 3 elements.

§ 5. Infinite Γ–monoid.

Now we shall determine the type of the infinite Γ–monoid in this paragraph.

Lemma 16. *An infinite Γ–monoid is a group.*

Proof. Let M be an infinite Γ–monoid, and G be this greatest group. Suppose that $G \subsetneqq M$, then G is finite by Theorem 1, and the difference monoid of M modulo G is finite by Theorem 3. Accordingly M is finite; this contradicts with the assumption. This shows that $G = M$.

As a result of Theorem 2, Lemmas 2 and 3, the structure of an infinite Γ-monoid is clarified in the following manner.

At first, we shall explain a "limit group of groups". There is given an increasing sequence of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_\gamma \subset \cdots$$

and isomorphisms ϕ_δ^γ of G_γ into G_δ $(\gamma < \delta)$ satisfying $\phi_\xi^\delta \phi_\delta^\gamma = \phi_\xi^\gamma$. Let G be the union of G_γ $(\gamma = 0, 1, 2, \cdots)$: $G = \bigcup_\gamma G_\gamma$ and let $\bar{G}$ be the quotient set of G obtained by identifying

$$x \in G_\gamma \quad \text{with} \quad y = \phi_\delta^\gamma(x) \in G_\delta.$$

The product xy of x and y in $\bar{G}$ is defined as the product of x and y in a certain group G_γ containing them. Then $\bar{G}$ is clearly a group. $\bar{G}$ is called a limit group of $\{G_\gamma;\ \phi_\delta^\gamma\}$.

Now, in an infinite Γ-monoid M there is a basic chain $\{[a_\gamma]\}$ such that $[a_\gamma]$ is a cyclic group of prime power order p^γ and

$$M = \bigcup_{\gamma=0}^{\infty} S_\gamma$$

where $\qquad S_\gamma = [a_\gamma], \quad a_0 = e, \quad a_\gamma = a_{\gamma+1}^p \qquad (\gamma = 0, 1, 2, \cdots).$

It is readily seen that M is a limit group of $\{S_\gamma;\ \phi_\delta^\gamma\}$ where ϕ_δ^γ is a mapping of each element of S_γ into itself in S_δ.

Conversely, if we are given cyclic groups S_γ of order p^γ $(\gamma = 0, 1, 2, \cdots)$, an isomorphism ϕ_δ^γ of S_γ into S_δ is uniquely determined and it holds $\phi_\xi^\delta \phi_\delta^\gamma = \phi_\xi^\gamma$. Accordingly we can consider the limit group of $\{S_\gamma;\ \phi_\delta^\gamma\}$. Then then sequence

$$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_\gamma \subset \cdots$$

is a full chain of power submonoids of M, because there is no power submonoid S_δ between S_γ and $S_{\gamma+1}$ $(\gamma = 0, 1, \cdots)$. Consequently, by Theorem 2, M is a Γ-monoid and

$$M = \bigcup_{\gamma=0}^{\infty} [a_\gamma]$$

where a_γ is a generator of S_γ, or $\{[a_\gamma]\}$ is a basic chain of M.

Theorem 4. *An infinite Γ-monoid is a limit group of cyclic groups S_γ $(\gamma = 0, 1, \cdots)$ of order p^γ where p is a prime number, and vice versa.*

Corollary. *An infinite Γ-monoid is isomorphic with the additive group E of modulo 1 as follows.*

$$E = \left\{ \frac{m}{p^n}\ ;\ m = 0, 1, 2, 3, \cdots, p^n - 1;\ n = 0, 1, 2, 3, \cdots \right\}.$$

§ 6. Finite Γ-monoids.

Finally we shall establish all types of finite Γ–monoids.

Lemma 17. *A finite Γ–monoid is a power monoid.*

Proof. The full chain of power monoids of a finite Γ–monoid M ceases at finite terms:

$$[a_0] \subset [a_1] \subset \cdots \subset [a_n] \quad \text{and} \quad M = \bigcup_{i=0}^{n} [a_i] .$$

Take any $x \in M$, then $x \in [a_t] \subset [a_n]$ for some $t \leqq n$. Hence $M \subset [a_n]$; we have $M = [a_n]$.

Since the greatest group G of a finite Γ–monoid M is a cyclic group of prime power order p^m, the types of M is limited to the three, because of Theorem 3,

(1) M is a power monoid of order p^m *i. e.*, M is a cyclic group,

(2) M is a power monoid of order p^m+1,

(3) M is a power monoid of order p^m+2,

where p^m is the order of G.

Hereafter we shall investigate the types of (2) and (3).

Lemma 18. *Let p be a prime number. A power monoid M of order p^m+1, whose greatest group G is of order p^m, is a Γ–monoid.*

Proof. Let a be a generator of M. It is not hard to see

$$M = \{a, a^2, a^3, \cdots, a^{p^m}, a^{p^m+1}\},$$

where $$a^2 = a^{p^m+2} \quad \text{and} \quad G = \{a^2, a^3, \cdots, a^{p^m+1}\}.$$

Since a submonoid containing a coincides with M, we see easily that M is a Γ–monoid.

As to (3), we divide the cases into the two: $p \neq 2$ and $p = 2$.

Lemma 19. *Let p be a prime number $\neq 2$. A power monoid M of order p^m+2, whose greatest group G is of order p^m, is a Γ–monoid.*

Proof. We may prove that a submonoid S containing a^2 is nothing but $\{a^2\} \cup G$. Given any $\mu_0 \geqq 3$,

$$2\nu \equiv \mu_0 \qquad (mod.\ p^m)$$

has a solution ν. This shows that all elements of G are generated by a^2. Hence $S = \{a^2\} \cup G$.

Lemma 20. *A power monoid M of order 2^m+2, whose greatest group is of order 2^m, is not a Γ–monoid.*

Proof. In order that the congruence equation

$$2\nu \equiv \mu_0 \qquad (mod.\ 2^m)$$

has a solution, μ_0 must be even. Let S be a power submonoid generated by a^2, then $S \frown G = \{a^4, a^6, \cdots, a^{2^m+2}\}$. It follows that S and G are incomparable.

Putting together Lemmas 18–20, we have

Theorem 5. *A finite monoid M is a Γ-monoid if and only if M is a power monoid of order n satisfying two conditions:*

(1) *the greatest group G of M is of prime power order p^m,*

(2) $p^m \leqq n \leqq p^m+1$ *if* $p=2$,

$p^m \leqq n \leqq p^m+2$ *if* $p \neq 2$.

Thus we have established all types of finite or infinite Γ-monoids.

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NOTES ON FINITE SEMIGROUPS AND DETERMINATION[0] OF SEMIGROUPS OF ORDER 4

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In the present paper we shall give some remarks about finite semigroups and shall determine all types of semigroups of order 4. The computation is performed by use of the elementary method [1][1], the results of semigroups of order 2, 3 [2], and our developed theories of semigroups. A general method of determination of finite semigroups (of an arbitrary order) is not yet found out.

§ 1. Unipotent semigroups.

1 Unipotent semigroups. In the previous paper [3] we argued some properties of finite unipotent semigroups. Furthermore we argued them from more general standpoint in another article [4] by Clifford and Miller's theory [5].

We mean by a zero-semigroup a unipotent semigroup whose idempotent is a two-sided zero 0. Apart from zero-semigroups of order n, unipotent semigroups of order n are determined in such a way as following, if zero-semigroups of order $m < n$ are all given.

A group G of order g, $g < n$, a zero-semigroup Z of order m where $m = n - g + 1$, and a homomorphism f of $M = G \cup Z'$ onto G determine uniquely a unipotent semigroup of order n, greatest group of which is G [4].

Z' symbols the set of all elements of Z except a zero, and M is the union of G and Z'.

All unipotent semigroups of order 4 other than zero-semigroups are $u-11 \sim u-19$[2], in which I. (2)[3] is the class of types $g = 2$, I. (3) types $g = 3$, and I. (4) groups.

2 Zero-semigroups. Let a be an element of a semigroup S. If there exists $x \in S$ such that $ax = a$, a is called an left-invariant (or l-invariant)

[0] This research was sponsored, in part, by MIKI-KORAKUKAI. See Addendum at the end.
[1] The number in the bracket [ ] shows the number of References appearing at the end.
[2] It is an individual number of a type in the table at the end.
[3] It is a number of a class of types in the same table.

element. Right-invariant (r–invariant) element is likewise defined. Here we denote by Z a finite zero-semigroup.

Lemma 1. *Z contains no l–invariant (r–invariant) element except* 0.

Proof. Suppose that there is an l–invariant element a different from 0. Set $X=\{x\,;\ x\in Z,\ ax=a\}$. It is seen that X is a subsemigroup of Z and does not contain 0; whence no idempotent lies in X because Z is a zero-semigroup. This conflicts with the fact that a finite semigroup contains at least one idempotent.

We introduce two orderings into Z: left ordering and right ordering. $a \underset{l}{\gtrsim} b$ means that either $a=b$ or $a=bx$ for some $x\in Z$, and $a \underset{r}{\gtrsim} b$ means that either $a=b$ or $a=yb$ for some $y\in Z$.

Lemma 2. *The two orderings are all partial orderings.*

Proof. Reflexivity and transitivity are clear. We shall prove anti-symmetry. If $a=bx$ and $b=ay$ for some x and y, then $a=a(yx)$. But, from Lemma 1, it follows that $a=0$ and so $a=b=0$. Similar as to right ordering.

Due to each of the two ordering, Z is a partly ordered set having 0 as greatest element. Now an element a is called an l–minimal element if $a \underset{l}{\gtrsim} b$ for no $b\neq a$. Likewise an r–minimal element is defined. Since Z is finite, minimal elements exist. Specially when a is least, a is called l–least (r–least) element.

Lemma 3. *If a is an l–minimal element, then a is also r–minimal, and vice versa.*

Proof. If a is not l–minimal, then $a=bx$ for some $b,x\in Z$; so $a \underset{r}{\gtrsim} x$, that is, a is not r–minimal. The proof of the converse is similar. We notice that a may be supposed to be distinct from 0, because the case of a trivial zero-semigroup $Z=\{0\}$ is out of consideration.

Lemma 4 *Let Z be a finite zero-semigroup. The following conditions are all equivalent.*

(i) *Z has an l–least element.*
(ii) *Z has an r–least element.*
(iii) *Z forms a chain with respect to the l–ordering.*
(iv) *Z forms a chain with respect to the r–ordering.*
(v) *Z is a power semigroup.*

Proof.

$$\text{(ii)} \rightleftarrows \text{(i)} \to \text{(v)} \begin{cases} \nearrow \text{(iii)} \to \text{(i)} \\ \searrow \text{(iv)} \to \text{(ii)}\,. \end{cases}$$

(i) $\rightleftarrows$ (ii): obvious by Lemma 3. (i) $\to$ (v). Let a be the l–least element. Every $x \neq a$ is written $x = ay$, that is, the v–order is $n-1$ where n is the d–order of Z. According to [6], Z is a power semigroup. (v) $\to$ (iii) and (v) $\to$ (iv) are easily proved. (iii) $\to$ (i) and (iv) $\to$ (ii) are clear.

We denote by $[a]$ the power subsemigroup generated by a:

$$[a] = \{a^i\,;\ i = 1, 2, \dots\}\,.$$

Now we define the third ordering in Z: $a \geqq b$ means that $[a] \subset [b]$. This is called the power-ordering (p–ordering).

Lemma 5. *$a \geqq b$ if and only if $a = b^n$ for some positive integer n.* (see [7])

Lemma 6. *The ordering $\geqq$ is a partial ordering.*

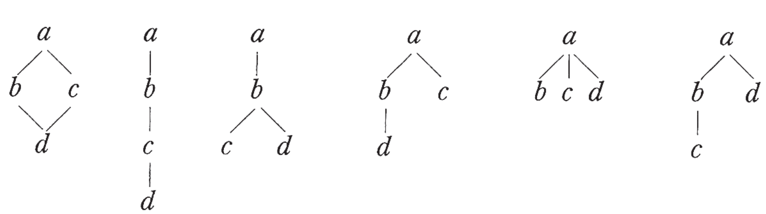
Proof. We shall show anti-symmetry. If $a \geqq b$ and $b \geqq a$ i. e., $b = a^n$ and $a = b^m$, then $a = a^{mn}$ which leads to $a = 0$ by Lemma 1, and hence $a = b = 0$.

Lemma 7. *$a \geqq b$ implies $a \underset{l}{\gtrsim} b$ and $a \underset{r}{\gtrsim} b$.*

Proof. By Lemma 5, $a = b^n = bb^{n-1} = b^{n-1}b$.

We shall construct, as an example, all types of zero-semigroups Z of order 4 by the aid of the above lemmas.

All types of partly ordered set of order 4, which has a greatest element, are shown as following.



0–1 0–2 0–3 0–4 0–5 0–4′

These become naturally semilattices.

Under consideration of Lemmas 3, 4 and 7, the following table designates all possible triple combinations chosen among them as l–, r– and p–orderings in Z and all types of Z deduced from the combinations. By Lemma 4, 0–1 cannot be taken as l–ordering (r–ordering).

The class [I].

l-ordering	0–5	0–2		0–4	0–4	0–3	0–3			0–4	0–4′
r-ordering	0–5	0–2		0–4	0–3	0–4	0–3			0–4′	0–4
p-ordering	0–5	0–1	0–2	0–4	0–4	0–4	0–3	0–4′	0–5	0–5	0–5
type of Z	*u*–1	*u*–2	*none*	*u*–3	*u*–4	*u*–4′	*u*–5 *u*–7 *u*–7′ *u*–8	*u*–6	*u*–9	*u*–10	*u*–10′

where *u*–4′, *u*–10′ are dually isomorphic with *u*–4, *u*–10 respectively.

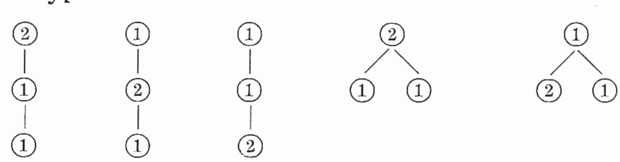
§ 2. Commutative semigroups.

According to [8], a finite commutative semigroup S is decomposed into the class sum of mutually disjoint unipotent subsemigroups and the quotient set forms a semilattce. Let L be a semilattice obtained in greatest decomposition of S by which $S=\bigcup_{i=1}^{n} S_i$.

By the types of L and S_i, all types of a non-unipotent commutative semigroup S of order 4 is classified into the following. Below, ⓘ symbols a unipotent commutative semigroup of order i ($i=1, 2, 3$)

(1) Semilattice,

(2) 2–1–1 type.



l–1 *l*–2 *l*–3 *l*–4 *l*–5 ,

(3) 2–2 type

l–6 ,

(4) 3–1 type

l–7 *l*–8 .

On the other hand, the types of ② and ③ are

② $\begin{vmatrix} A & B \\ B & A \end{vmatrix}$ U_2-1 $\begin{vmatrix} A & A \\ A & A \end{vmatrix}$ U_2-2 ,

③ $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix}$ U_3 1 $\begin{vmatrix} A & B & B \\ B & A & A \\ B & A & A \end{vmatrix}$ U_3-2 $\begin{vmatrix} A & B & A \\ B & A & B \\ A & B & A \end{vmatrix}$ U_3-3 $\begin{vmatrix} A & A & A \\ A & A & A \\ A & A & B \end{vmatrix}$ U_3-4 $\begin{vmatrix} A & A & A \\ A & A & A \\ A & A & A \end{vmatrix}$ U_3-5 .

Computating all types of S by elementary operations in each ease, we have

(1) Semilattice: VI. (1), i. e., c–1 ∼ c–5,

(2) 2–1–1 type: VI. (2),

② \ L	l–1	l–2	l–3	l–4	l–5
U_2–1	c–6	c–10	c–14	c–15	c–18
U_2–2	c–7 c–8 c–9	c–11 c–12	c–13	c–16 c–17	c–19

(3) 2–2 type: VI. (3),

lower \ upper	U_2–1	U_2–2
U_2–1	c–20 c–21	c–23 c–24
U_2–2	c–22	c–25 c–26

(4) 3–1 type: VI. (4),

L \ ③	U_3–1	U_3–2	U_3–3	U_3–4	U_3–5
l–7	c–27	c–28 c–29	c–30 c–31	c–32 c–33	c–34 c–35 c–36 c–37
l–8	c–38	c–39	c–40	c–42	c–41

§ 3 Non-commutative semigroups.

Generally a semigroup S is able to be decomposable to a commutative semigroup, i. e.,

$$S=\bigcup_{\alpha=1}^{k} S_{\alpha}, \qquad S_{\alpha}\cap S_{\beta}=\phi \qquad (\alpha \neq \beta),$$

and for α and β there is γ such that $S_{\alpha}S_{\beta}\subset S_{\gamma}$ and $S_{\beta}S_{\alpha}\subset S_{\gamma}$. It is proved that there exists a greatest decomposition of S to a commutative semigroup [9]. The meaning of greatest decomposition is due to [8].

Let C be a commutative semigroup obtained in greatest commutativity decomposition of S. We classify types of S into various classes according as the type of C.

1. *Commutativity-indecomposable semigroup.*

When C is composed of only one element, S is called commutativity-indecomposable (c–indecomposable).

Lemm 8. *If S is c–indecomposable, then $SS=S$.*

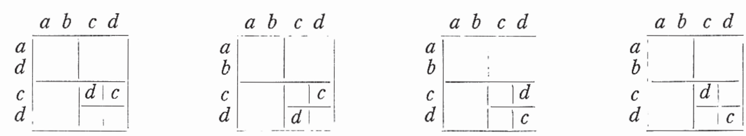
Proof. Suppose that $SS \subsetneqq S$.
Let $A = SS$ and $B = S - SS$, then $S = A \cup B$; this is a commutaitivity-decomposition.

Lemma 9. *If S is c–indecomposable semigroup of order* 4, *then S has no tow-sided ideal of order* 2, 3.

Proof. (i) The proof of having no ideal of order 3.
Suppose that S has an ideal A of order 3 and let $S = A \cup \{x\}$. Since $AS \subset A$ and $SA \subset A$, we have $x^2 = x$ by Lemma 8; whence the decomposition of S, $S = A \cup \{x\}$, gives commutativity.

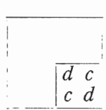
(ii) The proof of having no ideal of order 2.

Let $S = \{a, b, c, d\}$. Suppose that S has an ideal $\{a, b\}$. Since $SS = S$ and S has no ideal of order 3, the four cases are considered·

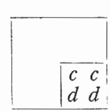


F–1 F–2 F–3 F–4

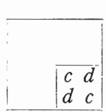
But, by an elementary theory,
we have



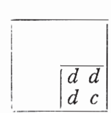
from F–1,



from F–2,



from F–3,



from F–4.

This shows that S is c–decomposable such that $S = \{a, b\} \cup \{c, d\}$, contradicting with the assumption. Hence S has no ideal of order 2.
Consequently we get

Theorem. *A c–indecomposable semigroup S of order* 4 *is completely simple* [10].

Proof. By Lemma 9, S has no ideal of order $\geqq 2$. Since a finite semigroup is simple, S is proved to be completely simple.

The theorem makes it possible to establish types of commutativity-indecomposable semigroup as II, $i-1$, $i-2$.

2. *c–decomposable semigroups.*
The types of C are as follows.

	A	B
A	A	A
B	A	B

D$_2$–1

	A	B
A	A	B
B	B	A

D$_2$–2

	A	B
A	A	A
B	A	A

D$_2$–3

D$_3$–1	D$_3$–2	D$_3$–3	D$_3$–4	D$_3$–5	D$_3$–6
A B C B C A C A B	A B B B A A B A A	A B A B A B A B A	A A A A A A A A B	A A A A A A A A A	A A A A B C A C B

D$_3$–7	D$_3$–8	D$_3$–9	D$_3$–10	D$_3$–11	D$_3$–12
A A A A B C A C A	A B A B A B A B C	A A A A B B A B C	A A A A B A A A C	A A A A A A A A C	A A A A B B A B B

After complicated computations we have the following result.

2–2 type: III,

C	D$_2$–1	D$_2$–2	D$_2$–3
S	(1) 2·2–1∼2·2–7	(2) 2·2–8	*none*

3–1 type: IV,

A	*a a a* *a a a* *a a a*	*a b c* *a b c* *a b c*	*a b a* *a b a* *a b a*
S	(1) 3·1–1 3·1–2	(2) 3·1–3∼3·1–8	(3) 3·1–9∼3·1–13

There is no type of *S* in other cases than above 3 types of A.

2–1–1 type: V,

C	D$_3$–1	D$_3$–2	D$_3$–3	D$_3$–4	D$_3$–5	D$_3$–6	D$_3$–7	D$_3$–8	D$_3$–9	D$_3$–10	D$_3$–11	D$_3$–12
S	*none*	*none*	*none*	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

We notice that dually isomorphic non-commutative semigroups are omitted in the table at the end. Thus we have obtained 194 types of semigroups of order 4.

Finally I express my heartfelt thanks to Mr. M. Yamamura, Mr. T. Akazawa and Mr. R. Shibata for their devotional works of the complicated computation.

Semigroups of order 4.

I. Unipotent

(1)

1

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	a

u–1

2

a	a	a	a
a	a	a	a
a	a	a	b
a	a	b	c

u–2

3

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	b

u–3

4

a	a	a	a
a	a	a	a
a	a	a	a
a	a	b	b

u–4

5

a	a	a	a
a	a	a	a
a	a	b	b
a	a	b	b

u–5

6

a	a	a	a
a	a	a	a
a	a	b	b
a	a	b	a

u–6

7

a	a	a	a
a	a	a	a
a	a	b	b
a	a	a	b

u–7

8

a	a	a	a
a	a	a	a
a	a	b	a
a	a	a	b

u–8

9

a	a	a	a
a	a	a	a
a	a	a	b
a	a	b	a

u–9

10

a	a	a	a
a	a	a	a
a	a	a	a
a	a	b	a

u–10

(2)

11

a	b	b	b
b	a	a	a
b	a	a	a
b	a	a	a

u–11

12

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	c

u–12

13

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	a

u–13

14

a	b	a	b
b	a	b	a
a	b	a	b
b	a	b	a

u–14

15

a	b	a	b
b	a	b	a
a	b	a	b
b	a	b	c

u–15

(3)

16

a	b	c	a
b	c	a	b
c	a	b	c
a	b	c	a

u–16

17

a	b	c	b
b	c	a	c
c	a	b	a
b	c	a	c

u–17

(4)

18

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

u–18

19

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

u–19

II. Commutativity-indecomposable

20

a	b	c	d
a	b	c	d
a	b	c	d
a	b	c	d

i–1

21

a	b	a	b
a	b	a	b
c	d	c	d
c	d	c	d

i–2

III. Decomposable, 2–2 type

(1)

22

a	b	a	a
a	b	a	a
a	b	c	d
a	b	c	d

2·2–1

23

a	b	a	a
a	b	b	b
a	b	c	d
a	b	c	d

2·2–2

24

a	b	a	b
a	b	a	b
a	b	c	d
a	b	c	d

2·2–3

25

a	a	a	a
a	a	a	a
a	b	c	d
a	b	c	d

2·2–4

26

a	a	a	a
a	a	b	b
a	a	c	d
a	a	c	d

2·2–5

27

a	b	a	a
a	b	a	a
a	b	c	c
a	b	d	d

2·2–6

28

a	b	a	a
a	b	b	b
a	b	c	c
a	b	d	d

2·2–7

(2)

29

a	b	c	d
a	b	c	d
c	d	a	b
c	d	a	b

2·2–8

IV. Decomposable, 3–1 type

(1)

30

a	a	a	a
a	a	a	b
a	a	a	c
a	a	a	d

3·1–1

31

a	a	a	a
a	a	a	b
a	a	a	a
a	a	c	d

3·1–2

(2)

32

a	b	c	a
a	b	c	a
a	b	c	a
a	b	c	d

3·1–3

33

a	b	c	a
a	b	c	b
a	b	c	a
a	b	c	d

3·14

34

a	b	c	a
a	b	c	b
a	b	c	c
a	b	c	d

3·1–5

35

a	b	c	d
a	b	c	d
a	b	c	d
d	d	d	d

3·1–6

36

a	b	c	a
a	b	c	a
a	b	c	a
a	b	c	a

3·1–7

37

a	b	c	a
a	b	c	a
a	b	c	b
a	b	c	a

3·1–3

(3)

38

a	b	a	a
a	b	a	a
a	b	a	c
a	b	a	d

3·1–9

39

a	b	a	a
a	b	a	b
a	b	a	c
a	b	a	d

3·1–10

40

a	b	a	a
a	b	a	b
a	b	a	a
a	b	c	d

3·1–11

41

a	b	a	a
a	b	a	a
a	b	a	a
a	b	c	d

3·1–12

42

a	b	a	b
a	b	a	b
a	b	a	b
a	b	c	d

3·1–13

V. Decomposable, 2–1–1 type

(1)

43

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	c

2·1·1–1

(2)

44

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	a

2

45

a	b	a	b
a	b	a	b
a	b	a	b
a	b	a	b

3

(3)

46

a	b	a	a
a	b	b	b
a	b	c	d
a	b	d	c

4

47

a	b	a	b
a	b	b	a
a	b	c	d
a	b	d	c

5

48

a	b	b	b
a	b	b	b
a	b	c	d
a	b	d	c

6

49

a	a	a	a
a	a	a	a
a	b	c	d
a	b	d	c

7

(4)

50

a	b	a	a
a	b	b	a
a	b	c	d
a	b	d	a

8

51

a	b	b	b
a	b	b	b
a	b	c	d
a	b	d	b

9

52

a	a	a	a
a	b	c	d
a	b	c	d
a	d	d	a

10

53

a	a	a	a
a	a	b	a
a	a	c	d
a	a	d	a

11

54

a	a	a	a
a	b	c	c
a	c	a	a
a	d	a	a

12

(5)

55

a	a	c	a
a	a	c	a
c	c	a	c
a	b	c	d

13

56

a	b	a	a
b	a	b	b
a	b	c	d
a	b	c	d

14

57

a	b	b	a
b	a	a	b
b	a	a	b
a	b	c	d

15

(6)

58

a	b	a	a
a	b	b	b
a	b	c	c
a	b	c	d

16

59

a	b	b	a
a	b	b	b
a	b	c	c
a	b	c	d

17

60

a	b	b	b
a	b	b	b
a	b	c	c
a	b	c	d

18

61

a	a	a	a
a	a	b	b
a	a	c	c
a	a	c	d

19

62

a	a	a	a
a	a	b	b
a	a	c	c
a	b	c	d

20

63

a	a	a	a
a	a	a	a
a	a	c	c
a	b	c	d

21

64

a	a	a	a
a	b	c	b
a	b	c	c
a	b	c	d

22

65

a	a	a	a
a	b	c	c
a	b	c	c
a	b	c	d

23

66

a	a	a	a
a	b	b	b
a	b	b	b
a	b	c	d

24

67

a	a	a	a
a	b	b	b
a	b	c	d
a	b	c	d

25

(7)

68

a	b	a	a
a	b	b	a
a	b	c	a
a	b	a	d

26

69

a	b	b	a
a	b	b	a
a	b	c	a
a	b	b	d

27

70

a	b	b	b
a	b	b	b
a	b	c	b
a	b	b	d

28

71

a	a	a	a
a	a	b	a
a	a	c	a
a	a	a	d

29

72

a	a	a	a
a	a	b	a
a	a	c	a
a	b	a	d

30

73

a	a	a	a
a	a	b	a
a	a	c	a
a	b	b	d

31

74

a	a	a	a
a	b	b	a
a	c	c	a
a	a	a	d

32

(8)

75

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	d

33

76

a	b	a	b
a	b	a	b
a	b	a	b
a	b	a	d

34

77

a	b	a	a
a	b	a	b
a	b	a	a
a	b	a	d

35

78

a	a	a	a
a	a	a	a
a	a	a	a
a	b	a	d

36

79

a	a	a	a
a	a	a	a
a	a	a	a
a	b	b	d

37

80

a	a	a	a
a	a	a	a
a	a	c	d
a	a	c	d

38

(9)

81

a	b	b	b
a	b	b	b
a	b	c	c
a	b	c	c

39

82

a	b	a	a
a	b	b	b
a	b	c	c
a	b	c	c

40

83

a	a	a	a
a	a	b	b
a	a	c	c
a	a	c	c

41

84

a	a	a	a
a	b	c	b
a	b	c	b
a	b	c	b

42

VI. Commutative, non-unipotent

(1) Semilattice

85

a	a	a	a
a	b	b	b
a	b	c	c
a	b	c	d

c–1

86

a	a	a	a
a	b	b	b
a	b	c	b
a	b	b	d

c–2

87

a	a	a	a
a	b	a	b
a	a	c	a
a	b	a	d

c–3

88

a	a	a	a
a	b	a	b
a	a	c	c
a	b	c	d

c–4

89

a	a	a	a
a	b	a	a
a	a	c	a
a	a	a	d

c–5

(2) Semilattice-decomposable 2–1–1 type

90

a	b	a	a
b	a	b	b
a	b	c	c
a	b	c	d

c–6

91

a	a	a	a
a	a	b	b
a	b	c	c
a	b	c	d

c–7

92

a	a	a	a
a	a	a	b
a	a	c	c
a	b	c	d

c–8

93

a	a	a	a
a	a	a	a
a	a	c	c
a	a	c	d

c–9

94

a	a	a	a
a	b	c	b
a	c	b	c
a	b	c	d

c–10

95

a	a	a	a
a	b	b	b
a	b	b	c
a	b	c	d

c–11

96

a	a	a	a
a	b	b	b
a	b	b	b
a	b	b	d

c–12

97

a	a	a	a
a	b	b	b
a	b	c	c
a	b	c	c

c–13

98

a	a	a	a
a	b	b	b
a	b	c	d
a	b	d	c

c–14

99

a	b	a	a
b	a	b	b
a	b	c	a
a	b	a	d

c–15

100

a	a	a	a
a	a	b	a
a	b	c	a
a	a	a	d

c–16

101

a	a	a	a
a	a	a	a
a	a	c	a
a	a	a	d

c–17

102

a	a	a	a
a	b	c	a
a	c	b	a
a	a	a	d

c–18

103

a	a	a	a
a	b	b	a
a	b	b	a
a	a	a	d

c–19

(3) Semilattice-decomposable 2–2 type

104

a	b	a	a
b	a	b	b
a	b	c	d
a	b	d	c

c–20

105

a	b	a	b
b	a	b	a
a	b	c	d
b	a	d	c

c–21

106

a	b	a	a
b	a	b	b
a	b	c	c
a	b	c	c

c–22

107

a	a	a	a
a	a	b	b
a	b	c	d
a	b	d	c

c–23

108

a	a	a	a
a	a	a	a
a	a	c	d
a	a	d	c

c–24

109

a	a	a	a
a	a	b	b
a	b	c	c
a	b	c	c

c–25

110

a	a	a	a
a	a	a	a
a	a	c	c
a	a	c	c

c–26

(4) Semilattice-decomposable 3–1 type

111

a	b	c	a
b	c	a	b
c	a	b	c
a	b	c	d

c–27

112

a	b	b	a
b	a	a	b
b	a	a	b
a	b	b	d

c–28

113

a	b	b	a
b	a	a	b
b	a	a	c
a	b	c	d

c–29

114

a	b	a	a
b	a	b	b
a	b	a	c
a	b	c	d

c–30

115

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	d

c–31

116

a	a	a	a
a	a	a	b
a	a	b	c
a	b	c	d

c–32

117

a	a	a	a
a	a	a	a
a	a	b	a
a	a	a	d

c–33

118

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	d

c–34

119

a	a	a	a
a	a	a	a
a	a	a	c
a	a	c	d

c–35

120

a	a	a	a
a	a	a	b
a	a	a	b
a	b	b	d

c–36

121

a	a	a	a
a	a	a	b
a	a	a	c
a	b	c	d

c–37

122

a	a	a	a
a	b	c	d
a	c	d	b
a	d	b	c

c–38

123

a	a	a	a
a	b	c	c
a	c	b	b
a	c	b	b

c-39

124

a	a	a	a
a	b	c	b
a	c	b	c
a	b	c	b

c-40

125

a	a	a	a
a	b	b	b
a	b	b	b
a	b	b	b

c-41

126

a	a	a	a
a	b	b	b
a	b	b	b
a	b	b	c

c-42

Addendum

Although all types of semigroups of order 4 were already found out by Mr. M. Yamamura & the writer in 1953, we have computed them once more by utilizing the new theories. We have heard from Prof. E. Hewitt, University of Washington, that Prof. G. E. Forsythe, Unviversity of California, is computing them by a very large electronic computer.

August, 1954.

References

[1] By the elementary method we mean the method of determination of semigroups by using Theorem 1 & 3 in [2].

[2] T. Tamura, Some remarks on semigroups and all types of semigroups of of order 2, 3, Jour. of Gakugei, Tokushima Univ., Vol. III, 1953, pp. 1–11.

[3] T. Tamura, On finite one-idempotent semigroups, Jour. of Gakugei, Tokushima Univ., Vol. IV, 1954, pp. 11–20.

[4] T. Tamura, Note on unipotent inversible semigroups, Kodai Math. Semi. Rep, No. 3, October, 1954. pp. 93–95.

[5] A. H. Clifford & D. D. Miller, Semigroups having zeroid elements, Amer. Jour. of Math., Vol. LXX, No. 1, 1948, pp. 117–125.

[6] See Theorem 9 at p. 19 in [3].

[7] See Lemma 2 in the paper: T. Tamura, On a monoid whose submonoids form a chain, in this Journal.

[8] T. Tamura, On decompositions of a commutative semigroup, Kodai Math. Semi. Rep. (unpublished).

[9] The proof will be dealed with in another paper.

[10] Rees, D., On semigroups, Proc. Cambridge Phil. Soc. Vol. 36, 1940, pp. 387–400.

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BIMODAL DISTRIBUTIONS[1)]

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These are practically frequently of use. It is said that K. Pearson answered it by the superposition of his fundamental unimodal specimen curves. However, as I could neither find such essays in the back numbers of Biometrika[2)] nor have the opportunity to search other references, *e.g.* Metron, Phil. Mag., Phil. Trans. &c., so it is tried to construct several new bimodal curves in the cases: (i) the distribution is in both sides unlimited, (ii) only in one side limited, (iii) in both sides limited.

§ 1. The differential equation[3)] of frequency curves in the most general form shall be given by

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_0+a_1x+a_2x^2+\cdots\cdots}{b_0+b_1x+b_2x^2+\cdots\cdots},$$

which, however, is too extensive to be treated here. To get simply bimodal curves, it is sufficient to assume that they become minimum at origin, and maximum at two other (oppositely lying) points, so that the required D. E. reduces merely to

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2}, \tag{1}$$

where the numerator is to have real roots of different signs besides 0.

(i) *The case, where the distribution is in both sides unlimited.* In this case the denominator in (1) must not have any real root, so that, for the sake of brevity, we may assume the denominator simply to be 1:

---

1) Although the present work is not so refined theoretically, the author aimed to utilize it as the stuff of exercise on mechanical computations for students: E. g. On the Decomposition of a bimodal Distribution into two normal Curves, T. Kudō and others, which, however, as has been not yet completed, would be published in the next number of this Journal.

2) Except the only one: Sui massimi delle curve dimorfiche, Dal Dr. Fernando de Helguero, Roma, Biometrika, vol. III (1904), p. 84,—which, however, does not go into details.

3) As well known, Pearson, starting from a problem of a game, adopts only the form $\frac{1}{y}\frac{dy}{dx}=\frac{a_0+x}{b_0+b_1x+b_2x^2}$, as the D. E. of his fundamental distributions.

$$\frac{1}{y}\frac{dy}{dx}=a_1x+a_2x^2+a_3x^3 .$$

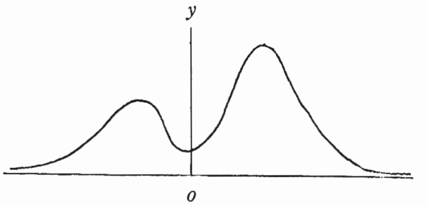

Fig. 1

We get, therefore,

$$\log\frac{y}{C}=\frac{1}{2}a_1x^2+\frac{1}{3}a_2x^3+\frac{1}{4}a_3x^4 ,$$

i.e. $y=C\exp\{ax^2+bx^3+cx^4\}$.

Or, upon taking the mean value $\bar{x}$ as origin, and writing $x-\bar{x}=u$, we have

$$y=y_0e^{\varphi(u)} \quad \text{with} \quad \varphi(u)=c_1u+c_2u^2+c_3u^3+c_4u^4 , \tag{2}$$

where, under assumption, $\varphi'(u)$ should have 3 real roots and $c_4<0$.

Now, in order to determine the constants in (2), we avail the moments formulas:

$$\int_{-\infty}^{\infty}y_0e^{\varphi(u)}du=\mu_0=1 , \quad \int_{-\infty}^{\infty}y_0u^ne^{\varphi(u)}du=\mu_n \quad (n=1,2,\cdots) , \tag{3}$$

in which numerical values of μ_n can be obtained from actual statistics, though the integrals themselves are not expressible in finite formes. So I make shift with the following treatment in a somewhat Pearson-like manner.

Firstly, integrating (3) by parts, we obtain

$$\mu_n=[y_0u^{n+1}e^{\varphi(u)}/(n+1)]_{-\infty}^{\infty}-\frac{y_0}{n+1}\int_{-\infty}^{\infty}(c_1u^{n+1}+2c_2u^{n+2}+3c_3u^{n+3}+4c_4u^{n+4})e^{\varphi(u)}du ,$$

in which the integrated parts become zero as $c_4<0$ by assumption, and the remaining integral can be expressed in terms of moments, so as

$$(n+1)\mu_n=-c_1\mu_{n+1}-2c_2\mu_{n+2}-3c_3\mu_{n+3}-4c_4\mu_{n+4} .$$

Putting here $n=0,1,2,3$ and observing that $\mu_0=1$, $\mu_1=0$, we get

$$\left.\begin{aligned} 0\ +2c_2\mu_2+3c_3\mu_3+4c_4\mu_4 &= -1 , \\ c_1\mu_2+2c_2\mu_3+3c_3\mu_4+4c_4\mu_5 &= 0 , \\ c_1\mu_3+2c_2\mu_4+3c_3\mu_5+4c_4\mu_6 &= -3\mu_2 , \\ c_1\mu_4+2c_2\mu_5+3c_3\mu_6+4c_4\mu_7 &= -4\mu_3 , \end{aligned}\right\} \tag{4}$$

from which the four unknowns c_1, c_2, c_3, c_4 can be determined. Substituted these values in (2) and (3), it gives

$$\int_{-\infty}^{\infty}\exp\{c_1u+c_2u^2+c_3u^3+c_4u^4\}du=\frac{1}{y_0} ,$$

whence by numerical computation the value of y_0 could be found. Since $c_4<0$ and the exponential tends rapidly to zero, we might execute the mechanical integration simply between $\pm L$ (numerically pretty large) instead of $\pm\infty$.

Example. A symmetrical distribution is given as in the second column of the following Table. Required to find the frequency curve.

x	y	x^2y	x^4y	x^6y
0	0.0690	0	0	0
± 1	0.0819	0.082	0.08	0.1
± 2	0.1210	0.484	1.94	7.7
± 3	0.1553	1.398	12.58	113.2
± 4	0.0952	1.523	24.37	389.9
± 5	0.0120	0.300	7.50	187.5
± 6	0.0001	0.004	0.13	4.7
sum	$1.0000=\mu_0$	$7.582=\mu_2$	$93.20=\mu_4$	$1406.2=\mu_6$

By reason of symmetry we may assume the distribution to be $y=y_0 \exp(c_2x^2+c_4x^4)$, and accordingly moments of odd order $=0$. Substituting the values of moments acquired from the above Table in (4), we find that $c_2=0.181$, $c_4=-0.010$ and directly $y_0=0.069$. Hence the required distribution is given by $y=0.069 \exp(0.18x^2-0.01x^4)$, roughly.

As done above, the actual moments μ_n are usually computed by summations, but to speak more exactly, they need Sheppard's corrections, as well known, and this is so, not only for $n=2$ and 4, but also for $n>4$. In general, if the ficticious and true moment of order n about y-axis are ν_n' and ν_n respectively, *i.e.*

$$\nu_n'=\sum_i f_i x_i^n, \quad \nu_n=\int_{-\infty}^{\infty} f(x)x^n dx,$$

we have, in the case that $y=f(x)$ highly osculates x-axis,

$$\nu_n=\nu_n'-\frac{w^2}{3!4}n(n-1)\nu_{n-2}-\frac{w^4}{5!4^2}n(n-1)(n-2)(n-3)\nu_{n-4}-\cdots\cdots,$$

where $w=$ breadth of class taken in summation; and thus

$$\nu_0=\nu_0'=1,\ \ \nu_1=\nu_1'=d,\ \ \nu_2=\nu_2'-\frac{w^2}{12},\ \ \nu_3=\nu_3'-\frac{w^2}{4}d,\ \ \nu_4=\nu_4'-\frac{w^2}{2}\nu_2-\frac{w^4}{80},$$

$$\nu_5=\nu_5'-\frac{5}{6}w^2\nu_3-\frac{w^4}{16}d,\ \ \nu_6=\nu_6'-\frac{5}{4}w^2\nu_4-\frac{3}{16}w^4\nu_2-\frac{w^6}{1792},$$

$$\nu_7=\nu_7'-\frac{7}{4}w^2\nu_5-\frac{7}{16}w^4\nu_3-\frac{w^6}{64}d,\ \ \nu_8=\nu_8'-\frac{7}{3}w^2\nu_6-\frac{7}{8}w^4\nu_4-\frac{w^6}{16}\nu_2-\frac{w^8}{2304},$$

$$\nu_9=\nu_9'-3w^2\nu_7-\frac{63}{40}w^4\nu_5-\frac{3}{16}w^6\nu_3-\frac{w^8}{256}d,\ \cdots\cdots.$$

So the higher the order of moment, the larger the correction. In particular, if the origin be the mean, ν_n and ν_n' become μ_n and μ_n', the moments about mean, respectively. Notwithstanding the above correction formulas hold the same and become even simpler, because $\mu_1=\mu_1'=d$ reduces to zero.

If we make Sheppard's correction in the preceding example, we obtain $\mu_2=7.499$, $\mu_4=89.47$, and $\mu_6=1293.0$ (though this is really of no use in the present case), so that the results become $c_2=0.210$, $c_4=-0.012$ and $y=0.069 \exp(0.21x^2-0.012x^4)$, thus pretty differ from those obtained before.

(ii) *The case, where the left handed side is limited, but the other side unlimited.* Assuming that the distribution extends from $x=-\gamma$ (negative) to $x=\infty$, the D. E. (1) can be written in the form

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_1x+a_2x^2+a_3x^3}{x+\gamma}\left(=\frac{\psi(x)}{x+\gamma}\right),\quad \gamma>0,\ \psi(-\gamma)\neq 0,\ a_3<0,\ a_1>0.$$

This yields after integration

$$y=k\exp\{ax+bx^2+cx^3\}\cdot(x+\gamma)^{-a\gamma},$$

where $a=a_1-a_2\gamma+a_3\gamma^2=\psi(-\gamma)/-\gamma\neq 0$,

$b=\frac{1}{2}(a_2-a_3\gamma)$, $c=\frac{a_3}{3}$ and $-a\gamma>-1$.[4]

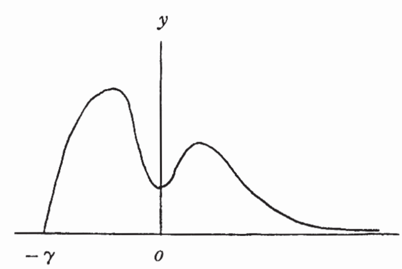

Fig. 2

Or, if we take $x=-\gamma$ as origin, and put $x=X-\gamma$, then the equation reduces to

$$\left.\begin{aligned}&y=KX^pe^{\varphi(X)},\quad \varphi(X)=c_1X+c_2X^2+c_3X^3,\\ &\text{where}\\ &c_1=a-2b\gamma+3c\gamma^2,\quad c_2=b-3c\gamma,\quad c_3=c<0\ \text{ and }\ p=-a\gamma>-1.\end{aligned}\right\}\quad(5)$$

Now taking the n-th moment about $X=0$, we obtain

$$\nu_n=K\int_0^\infty X^{n+p}e^{\varphi(X)}dX$$
$$=\left[\frac{K}{n+p+1}X^{n+p+1}e^{\varphi(X)}\right]_0^\infty-\frac{K}{n+p+1}\int_0^\infty X^{n+p+1}(c_1+2c_2X+3c_3X^2)e^{\varphi(X)}dX,$$

in which the integrated parts reduce to zero, and the remaining integral can be expressed in terms of moments of higher order, so as

$$(n+p+1)\nu_n+c_1\nu_{n+1}+2c_2\nu_{n+2}+3c_3\nu_{n+3}=0.$$

---

[4] The assumption $-a\gamma>-1$ is made, so that the integration at $x=-\gamma$ may be possible. When $a<0$, the curve really intersects x-axis at $x=-\gamma$; but if $a>0$ (yet $a<1/\gamma$), the negative root of $\frac{dy}{dx}=0$ goes out from the interval $(-\gamma,\infty)$, and there y becomes imaginary, so that the curve degenerates J-shaped, having $x=-\gamma$ as asymptote. Suchlike gives rise, when p or $q<0$ in (iii) below.

Putting here $n=0,1,2,3,4$, and in view of $\nu_0=1$, $\nu_1=d$ (mean), we get

$$\left.\begin{aligned}(p+1)\quad &+c_1 d+2c_2\nu_2+3c_3\nu_3=0\,,\\(p+2)d\ &+c_1\nu_2+2c_2\nu_3+3c_3\nu_4=0\,,\\(p+3)\nu_2&+c_1\nu_3+2c_2\nu_4+3c_3\nu_5=0\,,\\(p+4)\nu_3&+c_1\nu_4+2c_2\nu_5+3c_3\nu_6=0\,,\\(p+5)\nu_4&+c_1\nu_5+2c_2\nu_6+3c_3\nu_7=0\,.\end{aligned}\right\}\qquad(6)$$

These ν_n's can be expressed in terms of moments μ_n's about the mean $X=d\,(=\nu_1)$, all of which are obtainable from the given statistics:

$$\nu_n=\mu_n+n\mu_{n-1}d+\frac{n(n-1)}{2}\mu_{n-2}d^2+\cdots\cdots+\frac{n(n-1)}{2}\mu_2 d^{n-2}+d^n\,,$$

and thus

$$\left.\begin{aligned}&\nu_0=\mu_0=1,\quad \nu_1=d,\ (\mu_1=0),\quad \nu_2=\mu_2+d^2,\quad \nu_3=\mu_3+3\mu_2 d+d^3,\\&\nu_4=\mu_4+4\mu_3 d+6\mu_2 d^2+d^4,\quad \nu_5=\mu_5+5\mu_4 d+10\mu_3 d^2+10\mu_2 d^3+d^5,\\&\nu_6=\mu_6+6\mu_5 d+15\mu_4 d^2+20\mu_3 d^3+15\mu_2 d^4+d^6,\\&\nu_7=\mu_7+7\mu_6 d+21\mu_5 d^2+35\mu_4 d^3+35\mu_3 d^4+21\mu_2 d^5+d^7.\end{aligned}\right\}\qquad(7)$$

These being substituted in (6), we obtain five equations which involve five unknowns p, c_1, c_2, c_3 and d. If d be regarded as known parameter for a while, so (6) can be looked as simultaneous linear equations of c_1, c_2, c_3 and p. Therefore, on solving any four, say the latter four of (6), and substituting their values in the first, we get an equation of higher degree about d. If its root $d=d_0$ be adequately chosen, all numerical values of c_1, c_2, c_3, p could be computed. Lastly the coefficient K would be obtained from

$$\int_0^\infty X^p\exp\,(c_1X+c_2X^2+c_3X^3)dX=\frac{1}{K}$$

by means of mechanical integration. The distribution function (5) is thus completely determined.

(iii) *The case, where both sides are limited.* Let the ends of the distribution be $-\gamma(<0)$ and $\delta(>0)$. The D. E. (1) may be written as

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_1x+a_2x^2+a_3x^3}{(x+\gamma)(\delta-x)}\left(=\frac{\psi(x)}{(x+\gamma)(\delta-x)}\right),$$

where the quadratic $a_1+a_2x+a_3x^2$ should have two roots lying in $(-\gamma, o)$ and (o, δ) respectively, so that $a_1a_3<0$ and in fact

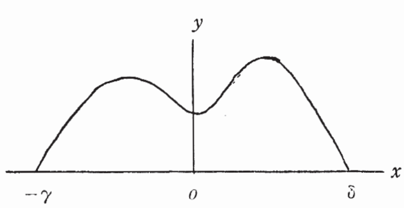

Fig. 3

$a_1 > 0$ and $a_3 < 0$, as seen below[5]. Solving the D. E. we get

$$y = K(x+\gamma)^p(\delta-x)^q \exp\left[\{(\gamma-\delta)a_3-a_2\}x-\frac{1}{2}a_3x^2\right] = f(x),$$

where

$$p = \frac{\psi(-\gamma)}{l}, \quad q = -\frac{\psi(\delta)}{l}, \quad \psi(x) = a_1x+a_2x^2+a_3x^3, \quad l = \gamma+\delta \text{ (breadth).}$$

Taking the left end $x = -\gamma$ as origin, and writing $x+\gamma = X$, the equation becomes

$$y = cX^p(l-X)^q \exp(aX+bX^2). \tag{8}$$

Its moment about $X = 0$ can be obtained as

$$\begin{aligned} \nu_n &= c\int_0^l X^{p+n}(l-X)^q \exp(aX+bX^2)dX \\ &= -\frac{c}{q+1}\Big[X^{p+n}(l-X)^{q+1}\exp(aX+bX^2)\Big]_0^l \\ &\quad + \frac{c}{q+1}\int_0^l \exp(aX+bX^2)X^{p+n}(l-X)^{q+1}\left\{\frac{p+n}{X}+a+2bX\right\}dX. \end{aligned}$$

Assumed that $p > -1$, $q > -1$[5], the integrated parts do vanish; and from the remaining integral, we have the following recurring formula

$$l(p+n)\nu_{n-1} = (n+1+p+q-al)\nu_n+(a-2bl)\nu_{n+1}+2b\nu_{n+2}.$$

On writing $\nu_0 = 1$, $\nu_1 = d$, and

$$lp = A, \quad l = B, \quad p+q-al = C, \quad a-2bl = D, \quad 2b = E, \tag{9}$$

the above yields

$$(A+nB)\nu_{n-1} = (n+1+C)\nu_n+D\nu_{n+1}+E\nu_{n+2}.$$

Putting $n = 1, 2, \cdots, 6$, we obtain the following six equations:

$$\begin{aligned} (A+B) &= (2+C)d+D\nu_2+E\nu_3, \\ (A+2B)d &= (3+C)\nu_2+D\nu_3+E\nu_4, \\ (A+3B)\nu_2 &= (4+C)\nu_3+D\nu_4+E\nu_5, \\ (A+4B)\nu_3 &= (5+C)\nu_4+D\nu_5+E\nu_6, \\ (A+5B)\nu_4 &= (6+C)\nu_5+D\nu_6+E\nu_7, \\ (A+6B)\nu_5 &= (7+C)\nu_6+D\nu_7+E\nu_8. \end{aligned} \tag{10}$$

[5] If we investigate more closely the sign of $\psi(x)$ and the expansion of $y = f(x)$ at origin &c., we see that, when p, q are both positive, and moreover if $a_1 > 0$, (so $a_3 < 0$), then the curve becomes really bimodal in $(-\gamma, \delta)$, but if $a_1 < 0$ $(a_3 > 0)$, only unimodal in $(-\gamma, \delta)$, whereas, if p, q are both negative, the curve degenerates U-shaped, and bi- or uni- antimodal according as $a_1 \gtrless 0$. If p, q be one positive and one negative, then the curve falls into a distorted J-shape.

Whence, by the same reasoning as done in (ii), all values of A, B, C, D, E and d can be determined, and in succession from (9) all values of l, p, b, a, q, and finally the value of c by the numerical computation of

$$\int_0^l X^p(l-X)^q \exp\{aX+bX^2\}dX = \frac{1}{c}.$$

If the definite integrals of $y, yX, \cdots$ could be expressed in finite forms, the method would become far more facile.

§ **2.** Since I could not find Pearson's essay on the construction of bimodal distribution by means of superposition of unimodals, a conjectured plan of his method should be described below.

(i) In the case, that both sides are unlimited, anyone would suppose immediately the superposition of two normal distributions. Nevertheless, the actual analysis is a pretty troublesome[6].

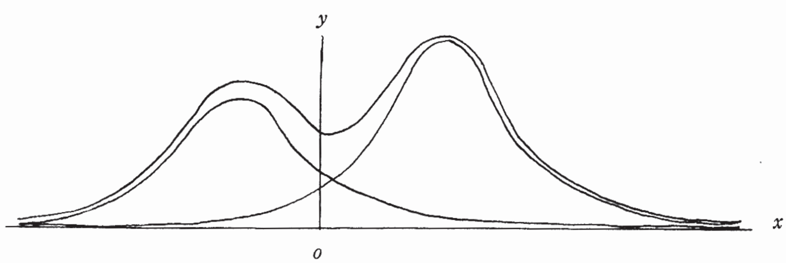

Fig. 4

The superposed frequency curve shall be

$$y = \frac{n_1}{\sqrt{2\pi}\,\sigma_1}\exp\left\{-\frac{(x-a_1)^2}{2\sigma_1^2}\right\} + \frac{n_2}{\sqrt{2\pi}\,\sigma_2}\exp\left\{-\frac{(x-a_2)^2}{2\sigma_2^2}\right\}, \qquad (1)$$

where $n_1+n_2=1$, if y is the probability density (or if y be the actual frequency, it shall be $n_1+n_2=N$, the actual total frequency).

To calculate the moment ν_n about origin, we write $t=(x-a_i)/\sigma_i$ $(i=1, 2)$ and integrate yx^n between $\pm\infty$. Making use of formulas

$$\int_{-\infty}^{\infty} t^n e^{-t^2}dt = \begin{cases} 0, & \text{when } n=\text{odd}, \\ \Gamma\left(\frac{n+1}{2}\right), & \text{when } n=\text{even}, \end{cases}$$

we get easily the following results:

---

6) Kudô and others, *l.c.*

$$\nu_0 = 1 = n_1 + n_2\,, \quad \nu_1 = n_1 a_1 + n_2 a_2 \ (= d = \text{total mean}),$$
$$\nu_2 = n_1(a_1^2+\sigma_1^2) + n_2(a_2^2+\sigma_2^2)\,, \quad \nu_3 = n_1 a_1(a_1^2+3\sigma_1^2) + n_2 a_2(a_2^2+3\sigma_2^2)\,,$$
$$\nu_4 = n_1(a_1^4+6a_1^2\sigma_1^2+3\sigma_1^4) + n_2(a_2^4+6a_2^2\sigma_2^2+3\sigma_2^4)\,,$$
$$\nu_5 = n_1 a_1(a_1^4+10a_1^2\sigma_1^2+15\sigma_1^4) + n_2 a_2(a_2^4+10a_2^2\sigma_2^2+15\sigma_2^4)\,.$$

From these six equations we must determine six unknowns:

$$n_1,\ n_2,\ a_1,\ a_2,\ \sigma_1,\ \sigma_2\,.$$

Specially for symmetrical distribution, the origin being the total mean, we have $n_1 = n_2 = \frac{1}{2}$, $a_2 = -a_1$, $\sigma_1 = \sigma_2$, and consequently $a_1^2+\sigma_1^2 = \mu_2$, $a_1^4+6a_1^2\sigma_1^2+3\sigma_1^4 = \mu_4$; whence we obtain

$$a_1^2 = \frac{1}{2}\sqrt{6\mu_2^2-2\mu_4}\,, \quad \sigma_1^2 = \mu_2 - \frac{1}{2}\sqrt{6\mu_2^2-2\mu_4}\,.$$

For the example treated in § 1, we have $\mu_2 = 7.499$, $\mu_4 = 89.47$, so that $a_1^2 = 6.294$, $a_1 = \pm 2.51$, $\sigma_1^2 = 1.205$, $\sigma_1 = 1.10$ and as the required function

$$y = \frac{2.20}{\sqrt{2\pi}}\left[\exp\frac{-(x-2.51)^2}{2.41} + \exp\frac{-(x+2.51)^2}{2.41}\right],$$

although this representation is not so good compared with that in § 1. To test the fitness more exactly, one ought to use χ^2– or ω^2–test.

(ii) If the distribution extends from $x = -\gamma$ to $x = \infty$, we may carry out the superposition of the curves from Pearson's type III

$$y = y_0\left(1+\frac{x}{\gamma}\right)^{c\gamma} e^{-cx}\,,$$

which have just alike ends. Or, on translating the origin into the left end, and writing $x+\gamma = X$,

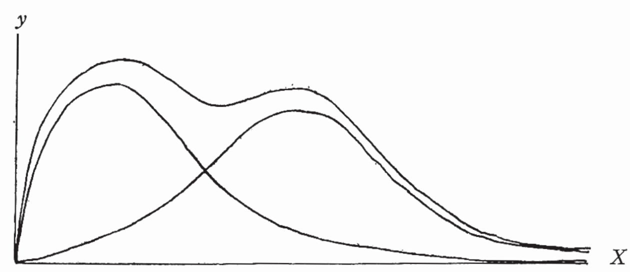

Fig. 5

$$y = kX^p e^{-cX} \quad (c > 0\,,\ p = c\gamma > -1)\,.$$

Hence the required bimodal curve shall be

$$y = kX^p e^{-cX} + k'X^{p'} e^{-c'X} \quad (c, c' > 0\,,\ p, p' > -1)\,. \tag{2}$$

The n–th moment of the first component about $X = 0$ is

$$k\int_0^\infty X^{n+p} e^{-cX} dX = k\int_0^\infty \left(\frac{t}{c}\right)^{n+p} e^{-t}\frac{dt}{c} \quad (cX = t)$$
$$= \frac{k}{c^{n+p+1}}\Gamma(n+p+1)\,.$$

Hence, if we put

$$\nu_0 = 1 = \frac{k\Gamma(p+1)}{c^{p+1}} + \frac{k'\Gamma(p'+1)}{c'^{p'+1}} = K + K',$$

the n-th moment of (2) becomes

$$\nu_n = \frac{(p+1)(p+2)\cdots\cdots(p+n)}{c^n} K + \frac{(p'+1)\cdots\cdots(p'+n)}{c'^n} K'.$$

By putting here $n = 0, 1, 2, \cdots, 6$, we obtain seven equations, which contain p, p', c, c' and K, K'. As done in (1.7), ν_n's can be expressed linearly in μ_n's, so that, on solving thus obtained seven equations, we can evaluate the seven unknowns, *i.e.* besides $\nu_1 = d$ the above six unknowns, and lastly k, k' from

$$k = \frac{Kc^{p+1}}{\Gamma(p+1)}, \quad k' = \frac{K'c'^{p'+1}}{\Gamma(p'+1)}$$

by use of the Table of gamma function.

Otherwise, if the given distribution make strong contact with X-axis at both ends, we may replace the foregoing by Pearson's type V, and thus consider

$$y = kX^{-q}\exp\{-\gamma/X\} + k'X^{-q'}\exp\{-\gamma'/X\}, \qquad (3)$$

where $\gamma, \gamma' > 0$ and q, q' are assumed to be sufficiently large, so that the moments of pretty higher order still may exist. Consequently, so far as $n < q-1$ is,

$$\nu_n = k\gamma^{n+1-q}\Gamma(q-n-1) + k'\gamma'^{n+1-q'}\Gamma(q'-n-1).$$

And if we put

$$\nu_6 = k\gamma^{7-q}\Gamma(q-7) + k'\gamma'^{7-q'}\Gamma(q'-7) = L + L',$$

the others can be written as

$$\begin{aligned}
\nu_5 &= L\gamma^{-1}(q-7) + L'\gamma'^{-1}(q'-7), \\
\nu_4 &= L\gamma^{-2}(q-7)(q-6) + L'\gamma'^{-2}(q'-7)(q'-6), \\
\nu_3 &= L\gamma^{-3}(q-7)(q-6)(q-5) + \cdots\cdots, \\
\nu_2 &= L\gamma^{-4}(q-7)(q-6)(q-5)(q-4) + \cdots\cdots, \\
\nu_1 &= L\gamma^{-5}(q-7)(q-6)(q-5)(q-4)(q-3) + \cdots\cdots, \\
\nu_0 &= L\gamma^{-6}(q-7)\cdots\cdots(q-2) + L'\gamma'^{-6}(q'-7)\cdots\cdots(q'-2).
\end{aligned}$$

Again, upon expressing ν_n's in μ_u's by (1.7), we can compute from these seven equations the seven unknowns $L, L', \gamma, \gamma', q, q'$ and $\nu_1 = d$, and lastly k, k' from

$$k = Lk^{q-7}/\Gamma(q-7), \quad k' = L\gamma'^{q'-7}/\Gamma(q'-7)$$

by means of the Table of gamma function.

(iii) In case that both ends are limited, we may refer to Pearson's type I, and assume

$$\left.\begin{aligned} y &= kX^{p}(l-X)^{q}+k'X^{p'}(l-X)^{q'}, \\ &(\text{all } p, q, p', q' > -1,\ l>0). \end{aligned}\right\} \qquad (4)$$

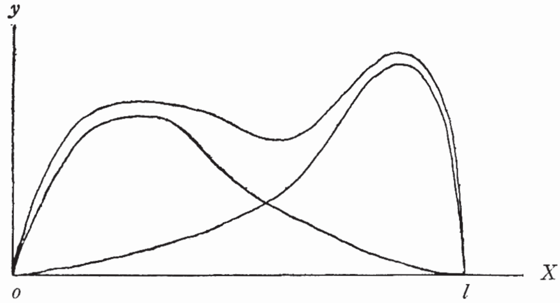

Fig. 6

The area under the curve is

$$\begin{aligned} \nu_0 = 1 &= k\int_0^l X^{p}(l-X)^{q}dX+k'\int_0^l X^{p'}(l-X)^{q'}dX \\ &= kl^{p+q+1}B(p+1, q+1)+k'l^{p'+q'+1}B(p'+1, q'+1), \end{aligned}$$

while the n-th moment

$$\begin{aligned} \nu_n &= k\int_0^l X^{n+p}(l-X)^{q}dX+k'\int_0^l X^{n+p'}(l-X)^{q'}dX \\ &= kl^{n+p+q+1}B(n+p+1, q+1)+k'l^{n+p'+q'+1}B(n+p'+1, q'+1). \end{aligned}$$

Hence, if A and A' be two components in ν_0, we obtain

$$\nu_n = \frac{(p+1)(p+2)\cdots\cdots(p+n)}{(p+q+2)\cdots\cdots(p+q+n+1)}l^n A+\frac{(p'+1)\cdots\cdots(p'+n)}{(p'+q'+2)\cdots\cdots(p'+q'+n+1)}l^n A'.$$

Here letting $n=0, 1, \cdots, 6$, we obtain, as before, seven equations containing seven unknowns, A, A', p, p', q, q' and $d(=\nu_1)$; whence all unknowns can be evaluated, and lastly k and k' from

$$k = A/l^{p+q+1}B(p+1, q+1), \quad k' = A'/l^{p'+q'+1}B(p'+1, q'+1)$$

by means of the Table of Beta function.

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ON THE MODIFIED COSINE FUNCTIONS

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In the previous note,[1] we had found, as a particular solution of the differential equation

$$\frac{d^2y}{dz^2}-\frac{2n}{z}\frac{dy}{dz}+\left(1+\frac{2n}{z^2}\right)y=0, \qquad (1)$$

an integral transcendental function

$$V_n(z)=\left[\frac{d^n}{d\zeta^n}(\cos\sqrt{1+2\zeta}\,z)\right]_{\zeta=0}=(-2z^2)^n\sum_{m=0}^{\infty}\frac{(-1)^m\,\underline{|m+n}}{\underline{|m}\,\underline{|2(m+n)}}z^{2m},$$
$$(n=0,1,2,\cdots). \qquad (2)$$

In the present note, we shall discuss the general solution of the differential equation, rather more generalized than (1), and the properties of the functions $V_n(z)$, which appear very similar to those of Bessel functions $J_n(z)$.

Although the differential equation (1) can be classified into the Bessel's equation in a broader sense,[2] yet its form is surely different from the ordinary Bessel equation $\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0$. So we venture to treat it below.

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1) This Journal, Vol. IV (1954), p. 39, Y. Watanabe and M. Nakamura, On the Partial Differential Equation of Paraboric Type with Constant Coefficients.

2) *Cf. e. g.* Whittaker and Watson, Modern Analysis, 3rd Edition, p. 203–4. Namely, the diffierential equation of the form

$$\frac{d^2u}{d\zeta^2}+\sum_{r=1}^{4}\frac{\frac{1}{2}-2\alpha_r}{\zeta-a_r}\frac{du}{d\zeta}+\left\{\sum_{r=1}^{4}\frac{\alpha_r\left(\alpha_r+\frac{1}{2}\right)}{(\zeta-a_r)^2}+\frac{A\zeta^2+2B\zeta+C}{\prod_{r=1}^{4}(\zeta-a_r)}\right\}u=0,$$

where $A=(\sum_{r=1}^{4}\alpha_r)^2-\sum_{r=1}^{4}\alpha_r^2+\frac{3}{2}\sum_{r=1}^{4}\alpha_r+\frac{3}{16}$ and B, C are constants, is called the generalized Lamé's equation. This differential equation has every point in the whole ζ-plane, except a_1, a_2, a_3, a_4, and ∞, as an ordinary point, these five points being all regular points with exponents α_r, $\alpha_r+\frac{1}{2}$ at a_r $(r=1, 2, 3, 4)$ and μ, $\mu+\frac{1}{2}$ at ∞. If we make two or more of these five singular points to tend to coincidence, we obtain thereby the so-called confluent equations. Among them, there is such a type which has only one regular, and only one irregular singularity, and else everywhere as ordinary behaves, and its type is called the (generalized) Bessel's equation. In this broader definition, no doubt, our present modified cosine function belongs to the (generalized) Bessel's functions. Therefore it will be more preferable to discuss more generally the Bessel equation in this broader sence. However we reserve this problem as a further task.

§ 1. We consider the differential equation of the form (1), but now n being not necessarily confined as a positive integer:

$$z^2\frac{d^2y}{dz^2}-2nz\frac{dy}{dz}+(z^2+2n)\,y=0\,,$$

whose singularity occurs regularly at $z=0$ but irregularly at $z=\infty$[3]. Let us find its formal solution

$$y=\sum_{\nu=0}^{\infty}a_\nu z^{\alpha+\nu}\,,$$

where the index α and coefficients a_ν's are to be determined. Substituting in the differential equation, we get

$$\sum_{\nu=0}^{\infty}\Big[\{(\nu+\alpha)(\nu+\alpha-1)-2n(\nu+\alpha)+2n\}\,a_\nu+a_{\nu-2}\Big]z^{\alpha+\nu}=0\,.$$

Equating coefficients of successive powers of z to zero, we obtain

$$(\nu+\alpha-2n)(\nu+\alpha-1)\,a_\nu+a_{\nu-2}=0 \qquad \begin{pmatrix}\nu=0,1,2,\cdots\\ a_{-1}=0,\quad a_{-2}=0\end{pmatrix}. \tag{3}$$

So for $\nu=0$

$$(\alpha-2n)(\alpha-1)\,a_0=0\,.$$

Hence the indicial equation has the roots $\alpha=2n$ and 1.

Firstly, taking $\alpha=2n$, we obtain the recurring formula

$$\nu(\nu+2n-1)\,a_\nu+a_{\nu-2}=0\,,\quad i.e,\qquad a_\nu=\frac{-a_{\nu-2}}{\nu(\nu+2n-1)}\,.$$

Hence, except $2n-1=$ *negative even*, $-2q$ say, we get for $\nu=2m$

$$a_{2m}=\frac{-a_{2m-2}}{2m(2m+2n-1)}=\frac{a_{2m-4}}{2^2m(m-1)(2m+2n-1)(2m+2n-3)}$$
$$=\cdots=\frac{(-1)^m\,\Gamma(2n+1)\,\Gamma(m+n+1)}{\underline{|m}\,\Gamma(n+1)\,\Gamma(2m+2n+1)}\,a_0\quad\begin{pmatrix}m=1,2,\cdots\\ n\neq -q+\frac{1}{2}\end{pmatrix},$$

which on putting $a_0=\dfrac{\Gamma(n+1)}{\Gamma(2n+1)}(-2)^n$ reduces to

$$a_{2m}=\frac{(-1)^{m+n}\,\Gamma(m+n+1)\,2^n}{\underline{|m}\,\Gamma(2m+2n+1)}\,.$$

For the sake of convenience[4] we may assume all $a_{2m+1}=0$ $(m=0,1,\cdots)$,

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3) Writing $z=\dfrac{1}{z_1}$, the equation (1) becomes $\dfrac{d^2y}{dz_1^2}+\dfrac{2(n+1)}{z_1}\dfrac{dy}{dz_1}+\dfrac{1+2nz_1^2}{z_1^4}y=0$, and thus the coefficient of y has a pole of order 4 at $z_1=0$, *i. e.* at $z=\infty$, hence there the equation is irregular (Unbestimmtheitsstelle).

4) Moreover, if n is neither negative integer nor 0, it is necessarily all $a_{2m+1}=0$, because, then we should have by (3) $2na_1=0$, as well as $(2m+1)\,2(m+n)\,a_{2m+1}+a_{2m-1}=0$, $(m=1,2,3,\cdots)$.

since they are quite independent of a_{2m} and surely satisfy the relation (3). Thus in case $2n-1 \neq -2q$ (even negative), we obtain, as the first particular solution, an infinite series $V_n(z)$:

$$y_1 = V_n(z) = (-2z^2)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|m}} \frac{\Gamma(m+n+1)}{\Gamma(2m+2n+1)} z^{2m}, \tag{4}$$

which is equal to $\left[\frac{d^n}{d\zeta^n}(\cos\sqrt{1+2\zeta}\, z)\right]_{\zeta=0}$, only when n is a positive integer or zero[5].

Next, taking the second root $\alpha = 1$, we obtain another recurring formula

$$\nu(\nu-2n+1)\, a_\nu + a_{\nu-2} = 0\,,$$

which can be availed for even ν if $2n-1$ be not positive even. Thus we get

$$a_{2m} = \frac{-a_{2m-2}}{2m(2m-2n+1)} = \cdots = \frac{(-1)^m a_0}{2^m \underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)}\,.$$

Hence, on putting again all $a_{2m+1} = 0$, we obtain, as the second particular solution,

$$y_2 = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{2^m \underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)} \qquad \left(n \neq q+\frac{1}{2}\right). \tag{5}$$

But if n is a positive integer, we have

$$\left[\frac{d^n}{d\zeta^n}\sin(\sqrt{1+2\zeta}\, z)\right]_{\zeta=0} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} z^{2m+1} \left[\frac{d^n}{d\zeta^n} \sum_{l=0}^{\infty} \binom{m+\frac{1}{2}}{l} 2^l \zeta^l\right]_{\zeta=0}$$

$$= \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} z^{2m+1} \sum_{l=n}^{\infty} \frac{\Gamma\left(m+\frac{3}{2}\right) 2^l}{\Gamma\left(m-l+\frac{3}{2}\right)} \frac{\zeta^{l-n}}{\underline{|l-n}}\right]_{\zeta=0}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} (2m+1)(2m-1)\cdots(2m-2n+3)\, z^{2m+1}. \tag{6}$$

Now, in order to equalize (5) and (6), we put the reserved constant

$$a_0 = \frac{(-1)^{n-1}\, \underline{|2n-2}}{2^{n-1}\, \underline{|n-1}} \qquad (n \geqslant 1)\,, \tag{7}$$

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5) Even though we take the Riemann-Liouville's fractional derivative, *e. g.* of order $n=1-\alpha$, $0<\alpha<1$, formally we get

$$D^n \cos\sqrt{1+2\zeta} z = DI^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{2m}} (1+2\zeta)^m = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{|2m}} \sum_{l=0}^{m} \frac{\underline{|m}\, 2^l}{\underline{|m-l}} DI^\alpha \frac{\zeta^l}{\Gamma(l+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{|2m}} \sum_{l=0}^{m} \frac{\underline{|m}}{\underline{|m-l}} \frac{2^l \zeta^{l+\alpha-1}}{\Gamma(l+\alpha)}\,.$$

To put here $\zeta=0$, it is no more than to obtain an absurd result

$$\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m}} z^{2m} \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)}\right]_{\zeta\to 0} = \text{indeterminate}.$$

then the expression (5) just coincides with (6). Really, by division of the corresponding summands, we get

$$\frac{a_0(-1)^m}{2^m \underline{|m}(2m-2n+1)\cdots(-2n+3)} \div \frac{(-1)^m}{\underline{|2m+1}}(2m+1)(2m-1)\cdots(2m-2n+3)$$

$$=\frac{a_0 \quad \underline{|2m+1}}{2\cdot4\cdots2m(2m+1)(2m-1)\cdots(2m-2n+3)(2m-2n+1)\cdots3\cdot1\cdot(-1)(-3)\cdots(-2n+3)}$$

$$=\frac{(-1)^{n-1}\underline{|2n-2}}{2^{n-1}\underline{|n-1}}\,\frac{2^{n-1}\underline{|n-1}}{(-1)^{n-1}\underline{|2n-2}}=1\,.$$

It is noteworthy to observe that the number of the linear factors in the denominator of summand in (5) is just m, so that it is available irrespectively whether n is a positive integer or not. However, in (6), the number of the linear factors in the numerator of summand is exactly n, and consequently (6) is not legitimate unless n is a positive integer. Hence, in general, we adopt the former and put

$$U_n(z)=\left(\frac{-1}{2}\right)^{n-1}\frac{\Gamma(2n-1)}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m+1}}{2^m\underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)} \tag{8}$$

which gives another particular solution, if $n-\frac{1}{2}\neq q$ (positive integer).

Thus, for any real n, except some trivial cases, we have obtained, as two particular solutions $V_n(z)$ and $U_n(z)$, generally independent of each other. In particular, when n is a positive integer, the two series $V_n(z)$ and $U_n(z)$ becomes

$$\begin{matrix}V_n(z)\\U_n(z)\end{matrix}=\left[\frac{d^n}{d\zeta^n}\begin{matrix}\cos\\\sin\end{matrix}(\sqrt{1+2\zeta}\,z)\right]_{\zeta=0}, \tag{9}$$

and might be called *modified cosine-* and *modified sine- functions* respectively. Surely they are independent of each other, as one is even function while the other is odd. Generally the general solution of (1) is given by

$$y=AV_n(z)+BU_n(z)\,, \tag{10}$$

where A and B are arbitrary constants. Of course, to say more exactly we have to examine several exceptional cases more minutely, and to secure valid solutions. The exceptional cases may occur when the difference of exponents $2n-1$ becomes an integer or zero. *E.g.* in the latter case we have $n=\frac{1}{2}$, and our series then become coincident:

$$U_{\frac{1}{2}}(z)=V_{\frac{1}{2}}(z)=\sum_{m=0}^{\infty}\frac{(-1)^{m+\frac{1}{2}}\Gamma\left(m+\frac{3}{2}\right)}{\underline{|m}\,\Gamma(2m+2)}2^{\frac{1}{2}}z^{2m+1}$$

$$=C\sum_{m=0}^{\infty}\frac{(-1)^m}{(\underline{|m})^2}\left(\frac{z}{2}\right)^{2m+1}=CzJ_0(z)\,,$$

thus it reduces to the Bessel function of order 0 multiplied with z. However, we reserve the discussion of all such special cases for future, and presently mainly confining to the case that n is a positive integer, and also rather laying stress upon $V_n(z)$, we proceed to deduce their properties.

§ 2. In the previous note[6)], it was seen that the relations

$$\frac{V_n(z)}{z^2}-\frac{V_n'(z)}{z}=V_{n-1}(z)\,, \quad i.e. \quad \left(\frac{1}{z}V_n(z)\right)'=-V_{n-1}(z) \tag{11}$$

and

$$V_n''(z)+V_n(z)=-2nV_{n-1}(z) \tag{12}$$

hold.

From (11) and (12) immediately follows

$$V_{n+1}(z)+(2n-1)\,V_n(z)+z^2V_{n-1}(z)=0\,, \tag{13}$$

and further this combined with (11) gives

$$V_n'(z)=\frac{2nV_n(z)}{z}+\frac{V_{n+1}(z)}{z}\,,$$

and also

$$\frac{d}{dz}\left(\frac{V_n(z)}{z^{2n}}\right)=\frac{V_{n+1}(z)}{z^{2n+1}}\,. \tag{14}$$

Since the form as the infinite series is invariable, whatever n may be, integral or fractional, all the above identities still hold even for non-positive integral n, so far as they exist.

Also if $\quad n>0\,, \quad \lim_{z\to 0} V_n(z)=o(1)=o(z^{2n-\varepsilon}) \qquad (\varepsilon>0)$

and if $\quad n>\frac{1}{2}\,, \quad \lim_{z\to 0} V_n'(z)=o(1)=o(z^{2n-1-\varepsilon})$.

§ 3. Now we shall prove the theorem that $V_n(z)$ with $n\geqslant 0$ has infinitely many real roots and moreover between any two consecutive real zeros of $V_n(z)$, there lies one and only one zero of $V_{n+1}(z)$. Since $V_n(z)$ is an even function, its real zero-points, if any besides $z=0$, should occur in pair of opposite signs with equal absolute value, so that we may only conceive its positive roots.

We prove the theorem by mathematical induction. At first for $V_0(z)=\cos z$ and $V_1(z)=-z\sin z$ the theorem is evident. Next let us assume that $V_n(z)=0$ and hence $V_n(z)/z^{2n}=0$ has infinitely many (discrete) roots.

6) loc. cip. p. 41.

In view of (14) together with Rolle's theorem, it follows that between each consecutive pair of zeros of $V_n(z)/z^{2n}$ there is at least one zero of $V_{n+1}(z)/z^{2n+1}$ Similarly, from (11) it follows that between each consecutive pair of zeros of $V_{n+1}(z)$ and hence of $V_{n+1}(z)/z$, there is at least one zero of $V_n(z)$. Therefore the theorem is true for $V_{n+1}(z)$, if it is true for $V_n(z)$. Hence it holds in general.

§ 4. To give another proof of the preceding theorem, we ready[7] an integral representation of $V_n(x)$: When $0<n<1$, $x=\frac{\pi\theta}{2}$, it holds that

$$V_n\left(\frac{\pi}{2}\theta\right)=\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)\pi^{2n-\frac{1}{2}}}{\Gamma(2n)\,2^n}\,\theta\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}\,dt\,. \tag{15}$$

To prove this, let us transform the integral

$$I=\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}\,dt$$

by putting $t=\theta\sqrt{s}$, as follows.

$$I=\int_0^1\frac{1}{\theta^{2(1-n)}(1-s)^{1-n}}\sum_{m=0}^\infty\frac{(-1)^m}{\underline{|2m}}\left(\frac{\pi}{2}\theta\right)^{2m}s^m\frac{\theta ds}{2\sqrt{s}}$$

$$=\frac{1}{2}\theta^{2n-1}\sum_{m=0}^\infty\frac{(-1)^m}{\underline{|2m}}\left(\frac{\pi}{2}\theta\right)^{2m}\int_0^1(1-s)^{n-1}s^{m-\frac{1}{2}}ds$$

But, as

$$B\left(n,m+\frac{1}{2}\right)=\frac{(2m-1)(2m-3)\cdots3\cdot\sqrt{\pi}}{(2n+2m-1)(2n+2m-3)\cdots(2n+1)}\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)},$$

so becomes

$$I=\frac{\Gamma(2n+1)\,2^{n-1}\sqrt{\pi}}{n\theta\Gamma\left(n+\frac{1}{2}\right)\pi^{2n}}\sum_{m=0}^\infty\frac{(-1)^m\Gamma(m+n+1)\,2^n}{\underline{|m}\,\Gamma(2m+2n+1)}\left(\frac{\pi}{2}\theta\right)^{2m+2n}$$

$$=\frac{(-1)^n\,2^n\Gamma(2n)\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)\theta\pi^{2n}}V_n\left(\frac{\pi}{2}\theta\right)=\frac{V_n\left(\frac{\pi}{2}\theta\right)}{C(n,\theta)},$$

whence (15) is proved.

As the coefficient $C(n,\theta)$ does not vanish in $0<\theta<\infty$, the vanishing of $V_n\left(\frac{\pi}{2}\theta\right)$ and that of the integral take place at the same time. Hence we have only to consider the change of the sign of the integral:

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7) Y. Watanabe, Über die Verschiebung der Nullstellen usw., this Journal, vol. III (1953), p. 16.

$$sgn\ V_n\left(\frac{\pi}{2}\theta\right) = sgn \int_0^\theta \frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt\,.$$

Divide the whole integration interval $0 < t < \infty$ at the points $t = 0, 1, 2, \cdots$. The function $\frac{1}{(\theta^2-t^2)^{1-n}}$ being monotonously increasing, if we put for any positive integer q

$$\int_{4q-2}^{4q} = v_{2q}\,, \quad \int_{4q}^{4q+2} = -v_{2q+1}\,,$$

it is clear that all v_p are positive and moreover v_p increases with p. Hence, if we write

$$\int_0^\theta \frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt = \int_0^{2p+\alpha} = \int_0^2 + \int_2^4 + \cdots + \int_{2p-2}^{2p} + \int_{2p}^{2p+\alpha}$$
$$= -v_1 + v_2 - \cdots + (-1)^p v_p + (-1)^p v_p'\,,$$

then

$$sgn\ V_n\left(\pi\left(p+\frac{\alpha}{2}\right)\right) = sgn\left[-v_1+v_2-v_3+\cdots+(-1)^p v_p + (-1)^p v_p'\right],$$

where $\quad 0 < v_1 < v_2 < \cdots < v_p\,,$ and also $v_p' \geqq 0$ if $0 \leqq \alpha \leqq 1$.

Therefore, according as $\quad p = even = 2q \quad$ or $\quad p = odd = 2q+1$,

$$sgn\ V_n\left(\pi\left(2q+\frac{\alpha}{2}\right)\right) = sgn\left[v_{2q}' + (v_{2q}-v_{2q-1}) + \cdots + (v_2 - v_1)\right] = +\,,$$

$$sgn\ V_n\left(\pi\left(2q+1+\frac{\alpha}{2}\right)\right) = sgn\left[-v_{2q+1}' - (v_{2q+1}-v_{2q}) - \cdots - (v_3-v_2) - v_1\right] = -\,.$$

Thus the change of sign of $V_n(x)$ in $0 < x < \infty$ happens an infinitely many times. The result just proved is obtained for the case $0 < n < 1$. However, it can be proved for the case $1 < n < 2$, $2 < n < 3, \cdots$, in the same way as done in § 3.

§ 5. Now we shall prove an integral theorem, which resembles to that of Lommel in regard to Bessel function. Let α and β be some different parameters $\neq 0$. Writing $z = \alpha x$ in (1), we have

$$\frac{d^2y}{dx^2} - \frac{2n}{x}\frac{dy}{dx} + \left(\alpha^2 + \frac{2n}{x^2}\right)y = 0\,,$$

one solution of which is obviously $V_n(\alpha x)$ and consequently

$$\frac{d^2V_n(\alpha x)}{dx^2} - \frac{2n}{x}\frac{dV_n(\alpha x)}{dx} + \left(\alpha^2 + \frac{2n}{x^2}\right)V_n(\alpha x) = 0\,.$$

Similarly

$$\frac{d^2V_n(\beta x)}{dx^2} - \frac{2n}{x}\frac{dV_n(\beta x)}{dx} + \left(\beta^2 + \frac{2n}{x^2}\right)V_n(\beta x) = 0\,.$$

Multiplying the former by $V_n(\beta x)$ and the latter by $V_n(\alpha x)$ respectively, and then subtructing side by side, we get

$$\frac{du}{dx}-\frac{2n}{x}u+(\alpha^2-\beta^2)\,V_n(\alpha x)V_n(\beta x)=0\,,$$

where

$$u=V_n(\beta x)\frac{dV_n(\alpha x)}{dx}-V_n(\alpha x)\frac{dV_n(\beta x)}{dx}\,.$$

Multiplying the differential equation just obtained by x^{-2n}, and integrating, it yields

$$\left[\frac{u}{x^{2n}}\right]_0^1=-(\alpha^2-\beta^2)\int_0^1\frac{1}{x^{2n}}\,V_n(\alpha x)V_n(\beta x)\,dx\,.$$

Since $\lim\frac{V_n(z)}{z^{2n}}=$ *finite*, and $\frac{dV_n(z)}{dz}$ vanishes at $z=0$ for $n>\frac{1}{2}$, so also $\frac{u}{x^{2n}}$ vanishes at $x=0$ (and this is also true for $n=0$, because of $V_0'(z)=-\sin z$). Thus the integrated part reduces to

$$\left[\frac{u}{x^{2n}}\right]_0^1=V_n(\beta)V_n'(\alpha)-V_n(\alpha)V_n'(\beta)\,.$$

Consequently we have (at least, when $n=$ *positive integer* or 0)

$$\int_0^1\frac{V_n(\alpha x)\,V_n(\beta x)}{x^{2n}}\,dx=\frac{1}{\alpha^2-\beta^2}\{V_n(\alpha)V_n'(\beta)-V_n(\beta)V_n'(\alpha)\}\qquad(\alpha\neq\beta)\,.\quad(16)$$

If we make β tend to α, the right-handed side of (16) becomes an indeterminate form $\frac{0}{0}$. However, on using l'Hospital's rule, and referring to (1) and (11), we can easily find the limiting value to be

$$\int_0^1\frac{V_n(\alpha x)^2}{x^{2n}}\,dx=\frac{1}{2\alpha}\{V_n'(\alpha)^2-V_n(\alpha)V_n''(\alpha)\}=\frac{1}{2\alpha}\{V_n'(\alpha)^2+V_n(\alpha)^2+2nV_n(\alpha)V_{n-1}(\alpha)\}$$

$$=\frac{1}{2\alpha}\left\{\left(1+\frac{1}{\alpha^2}\right)V_n(\alpha)^2+\alpha^2V_{n-1}(\alpha)^2+2(n-1)\,V_n(\alpha)V_{n-1}(\alpha)\right\}.\qquad(17)$$

§ 6. By use of the foregoing theorem, we can prove that $V_n(z)=0$ has really real roots only. For, in the integral theorem (16), *i.e.*

$$(\alpha^2-\beta^2)\int_0^1\frac{V_n(\alpha x)V_n(\beta x)}{x^{2n}}\,dx=V_n(\alpha)V_n'(\beta)-V_n(\beta)V_n'(\alpha)\,,$$

let $\alpha=\xi+i\eta$ be any roots of $V_n(z)=0$, then $\beta=\xi-i\eta$ should be so also, because the expansion (2) of $V_n(z)$ has only real coefficients. Accordingly

$$\left[(\xi+i\eta)^2-(\xi-i\eta)^2\right]\int_0^1\frac{V_n((\xi+i\eta)x)\,V_n((\xi-i\eta)x)}{x^{2n}}\,dx=0\,.$$

Here the integrand has the form

$$\frac{1}{x^{2n}}(P+iQ)(P-iQ)=\frac{P^2+Q^2}{x^{2n}}\geqslant 0,$$

and P^2+Q^2 cannot be 0 throughout any subinterval. Therefore the above integral is surely positive. Hence the multiplied factor $4\xi\eta i$ must vanish, so that $\xi=0$ or $\eta=0$, *i.e.* the root must be pure imaginary or pure real. But it is evident that

$$V_n(\pm\eta i)=(-1)^n(\eta i)^{2n}\sum_{m=0}^{\infty}\frac{(-1)^m\,\lfloor m+n}{\lfloor m\,\lfloor 2(m+n)}2^n(\eta i)^{2m}$$

$$=\eta^{2n}\sum_{m=0}^{\infty}\frac{\lfloor m+n}{\lfloor m\,\lfloor 2(m+n)}2^n\eta^{2m}>0,$$

Hence there is no pure imaginary root. Therefore any possible roots should be purely real, and really they exist as already shown in §§ 3, 4.

§ 7. We shall expand an arbitrary function $f(x)$, which is $L(0,1)$, in a series of $V_n(z)$ in the form

$$f(x)=\sum_{r=1}^{\infty}A_r V_n(\lambda_r x)^{8)},\tag{18}$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_r<\cdots$ denote real positive roots of $V_n(z)=0$. To determine a coefficient A_s, we multiply both members of (18) by $V_n(\lambda_s x)/x^{2n}$ and integrate from $x=0$ to $x=1$. Then by virtue of (16) and (17) we obtain

$$\int_0^1\frac{V_n(\lambda_s x)f(x)}{x^{2n}}dx=\int_0^1\sum_r=\sum_r A_r\int_0^1\frac{1}{x^{2n}}V_n(\lambda_r x)V_n(\lambda_s x)\,dx$$

$$=\frac{1}{2}A_s\lambda_s\{V_{n-1}(\lambda_s)\}^2.$$

Hence

$$A_s=2\int_0^1\frac{1}{x^{2n}}V_n(\lambda_s x)f(x)\,dx\Big/\lambda_s\{V_{n-1}(\lambda_s)\}^2,\qquad (s=1,2,3,\cdots).\tag{19}$$

For instance, if $n=1$, we get

$$A_s=-2\int_0^1\frac{1}{x^2}s\pi x\sin(s\pi x)\cdot f(x)\,dx\Big/s\pi(\cos s\pi)^2$$

$$=-2\int_0^1\frac{1}{x}\sin(s\pi x)f(x)\,dx.$$

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8) We have tacitly assumed that $f(x)$ is coutinuous throughout the interval (0, 1). It can be proved more rigorously in just the same manner as shown by Hobson (Proc. London Math. Soc. 2, vol. VII, 1909, p.p. 387-8, or Watson, Theory of Bessel functions 1922, p. 591.), that if $f(x)$ is absolutely integrable and of bounded variation in (0, 1), then the series is convergent and its sum is $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

§ 8. Lastly we shall prove that the two functions $V_n(z)$ and $U_n(z)$ are connected by the relation

$$U_n(z)V_{n-1}(z)-U_{n-1}(z)V_n(z)=z^{2n-1} \qquad (n \geqslant 1)\,. \tag{20}$$

For, since $U_n(z)$ and $V_n(z)$ satisfy the differential equation (1), we have

$$z^2U_n''(z)-2nzU_n'(z)+(z^2+2n)\,U_n(z)=0\,,$$
$$z^2V_n''(z)-2nzV_n'(z)+(z^2+2n)\,V_n(z)=0\,,$$

and whence

$$z^2(U_n''(z)V_n(z)-U_n(z)V_n''(z))-2nz(U_n'(z)V_n(z)-U_n(z)V_n'(z))=0\,,$$

that is

$$\frac{d}{dz}(U_n'(z)V_n(z)-U_n(z)V_n'(z))=\frac{2n}{z}(U_n'(z)V_n(z)-U_n(z)V_n'(z))\,.$$

On integrating we get

$$U_n'(z)V_n(z)-U_n(z)V_n'(z)=Cz^{2n}\,.$$

Substituting in the left handed side the value (11) of $V_n'(z)$ and similar one about $U_n'(z)$, which can be easily shown from the expansion (6) or (8), we get

$$U_n(z)V_{n-1}(z)-U_{n-1}(z)V_n(z)=Cz^{2n-1}\,.$$

To determine the value of C, we have only to find

$$\lim_{z\to 0}\left[\frac{U_n(z)}{z}\frac{V_{n-1}(z)}{z^{2(n-1)}}-U_{n-1}(z)\frac{V_n(z)}{z^{2n-1}}\right];$$

But it is easy to see that the limiting value becomes 1, by means of (4) (6) and (8), which proves (20).

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ON GENERALIZED DEVELOPMENT PROJECTIONS

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Introduction

To get a method of map projections is to be regarded to construct a one to one corespondence between a point on the globular surface and a point on the given plane.

After all, various kinds of problems of map projections are reduced to the discussions of mapping functions. Under geometrical interpretations, there are two methods of map projections, the one is to project the surface to a developable surface, and then develope it on a plane; the other is to project to a plane directly. Under analytical considerations there is no distinction between these two methods, but by the former, we have oftenly more superior distortionless projections for simpler mapping functions; so the conical or cylindrical projections have been applied for a long time. In this paper we wish to consider the generalization of development projections. As their computations are not so simple, it is not always prospective to get some practical ones usually; but in the special cases, there are some expectations to get useful projections by these methods. Then we wish to explain for general theories on the generalized projections.

§ 1. Isometric Coordinate Systems on a Tangent Surface of a Given Curve.

Put the equations of a space curve to

$$x^i = x^i(s) \quad (i = 1, 2, 3) \qquad \cdots\cdots\cdots\cdots (1)$$

where the parameter s means the arc length of the curve measured from a given point.

Then the equations of a tangent line at a point on the curve are given by

$$X^i = x^i(s) + t\alpha^i(s)\,^{1)}, \qquad \cdots\cdots\cdots\cdots (2)$$

where X^is mean the current coordinates and t is a parameter which means the distance from the tangent point to any point on the tangent line, and α^is mean the direction cosines of the tangent line.

If we consider s and t two independent variables in the equations (2), (2) represents the tangent surface of the curve (1). To obtain the fundamental magnitudes of (2), we consider the equations

$$\frac{d\alpha^i}{ds} = \kappa\beta^i\,, \qquad \sum \alpha^i\beta^i = 0\,, \tag{3}$$

where $\beta^i s$ mean the direction cosines of the principal normal of the curve (1), and κ means the curvature of the curve at the tangent point on it. Differentiating (2), with respect to s and t, we have

$$\frac{\partial X^i}{\partial s} = \alpha^i + t\kappa\beta^i\,, \quad \frac{\partial X^i}{\partial t} = \alpha^i\,,$$

then we have

$$\sum\left(\frac{\partial X^i}{\partial s}\right)^2 = \sum \alpha^{i2} + 2t\kappa\sum\alpha^i\beta^i + t^2\kappa^2\sum\beta^{i2} = 1 + t^2\kappa^2\,,$$

$$\sum\left(\frac{\partial X^i}{\partial t}\right)^2 = 1\,, \quad \sum\left(\frac{\partial X^i}{\partial s}\right)\left(\frac{\partial X^i}{\partial t}\right) = 1\,.$$

Then the line element of (2) is given by

$$ds^2 = dt^2 + 2dt\,ds + (1+t^2\kappa^2)\,ds^2\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots \tag{4}$$

where κ is a function of s.

In the equations (4) t and s are not the parameters for isometric coordinate systems, so we wish to get an isometric coordinate system on it from the equation (4). Deforming (4), we have

$$ds^2 = \{dt + (1+it\kappa)ds\}\{dt + (1-it\kappa)ds\}$$

then we put

$$dt + (1+it\kappa)ds \text{ to } pd\lambda\,, \quad \text{and} \quad dt + (1-it\kappa)ds \text{ to } qd\mu\,. \ \cdots\cdots \tag{5}$$

In (5) p and q mean the integrating factors of the expression, so we have from the integration of (5),

$$\lambda = \left(\frac{1}{\kappa}\tan^{-1} t\kappa + s\right) + \frac{i}{2\kappa}\log(1+t^2\kappa^2)$$

$$\mu = \left(\frac{1}{\kappa}\tan^{-1} t\kappa + s\right) - \frac{i}{2\kappa}\log(1+t^2\kappa^2)$$

Putting the real and imaginary parts of the above to x and y we have

$$x = \frac{1}{\kappa}\tan^{-1} t\kappa + s\,, \quad y = \frac{1}{2\kappa}\log(1+t^2\kappa^2)\,. \quad \cdots\cdots\cdots\cdots\cdots \tag{6}$$

Then the equation (4) is reduced to

$$ds^2 = (1+t^2\kappa^2)(dx^2+dy^2)\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots \tag{7}$$

In (7) parameters x and y are evidently isometric.

Then we have easily some conformal projections between the globular surface and the surface (2), applying the theories of functions of a complex variable. Developing the surface (2) to a plane, we have some new conformal projections of the earth.

§ 2. Envelope along a Given Curve on the Surface.

Put the equations of a given surface to

$$x = x(u\,v) \quad y = (u\,v) \quad z = z(u\,v)\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

then the equation of the tangent plane at the point $(x_0 y_0 z_0)$ on it is expressible to

$$\begin{vmatrix} X-x_0 & Y-y_0 & Z-z_0 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = 0\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (9)$$

where X, Y, Z, are current coordinates and x_u, y_v, etc. are partial derivatives of x and y with respect to u and v.

Specially, if (8) represent a surface of revolution, they are given by

$$x = p(v)\cos u \quad y = p(v)\sin u \quad z = q(v)\,, \qquad \cdots\cdots\cdots\cdots (10)$$

in (10) $p(v)$, $q(v)$ are arbitrary functions of v.

Then the equation (9) is reducible to

$$X\dot{q}\cos u + Y\dot{q}\sin u - Z\dot{p} = (p\dot{q} - \dot{p}q)\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (11)$$

Similarly, if the equations (8) represent a sphere, they are given by

$$x = \sin v \cos u \quad y = \sin v \sin u \quad z = \cos v\,, \qquad \cdots\cdots\cdots\cdots (12)$$

then (9) is reducible to

$$X\cos u\cos v + Y\sin u\cos v + Z\sin v = 1\,. \qquad \cdots\cdots\cdots\cdots (13)$$

If we consider v as a function of u in (11) and (13), they are equations of one parameter family of planes. Then these planes decide an envelope, of course, it is a developable surface.

If we can put the equation (11) or (13) to the form (2), it is able to determine an isometric coordinate system on it by the method explained in § 1.

To do so, we must determine the equation of edge of regression of the surface.

If the equation of family of planes involving a parametes t is given by

$$a(t)x+b(t)y+c(t)z+d(t)=0\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (14)$$

the equation of edge of regression of the family is obtained by solving x, y, z as functions of t from (14) and the followings,

$$\dot{a}(t)x+\dot{b}(t)y+\dot{c}(t)z+\dot{d}(t)=0 \qquad \ddot{a}(t)x+\ddot{b}(t)y+\ddot{c}(t)z+\ddot{d}(t)=0\,,$$

where $\dot{a}(t)$, $\ddot{a}(t)$, etc. mean the first and second derivatives of the coefficients with respect to t.

In this paper, for convenience of computations, we shall start from the equation (13).

From (13), describing the equations corresponding to the above three equations, we have,

$$\begin{aligned} &X\cos u\ \text{cus}\ v+Y\sin u\cos v+Z\sin v=1\\ &X(\sin u\cos v+\cos u\ \text{sins}\ v.v')+Y(-\cos u\cos v+\text{sins}\ u\ \text{sins}\ v.v')-Z\cos v.v'=0\\ &X(2\sin u\ \text{sins}\ v.v'-\cos u\ \text{sins}\ v.v'')+Y(-2\cos u\ \text{sins}\ v.v'-\sin u\ \text{sins}\ v.v'')\\ &\quad +Z(\cos v.v''+\text{sins}\ v)=v'^2+1\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (15)\end{aligned}$$

where v', v'' mean the first and second derivatives of v with respect to u.

Solving (15) with respect to x, y, z, we have

$$X=\frac{\sin u.v'(\cos^2 v+v'^2)+\cos u\cos v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')}$$

$$Y=\frac{-\cos u.v'(\cos^2 v+v'^2)+\sin u\cos v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')}$$

$$Z=\frac{(\cos^2 v+v'^2)+\sin v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (16)$$

As the equation (16) involving a parameter u, they represent the edge of regression of envelope of the family of planes (13).

Introducing m from the equation

$$\sin v.v'^2+\cos v.v''=m(\cos^2 v+v'^2)\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (17)$$

the equations (16) are reducible to

$$X=\frac{\sin u.v'+m\cos u\cos v}{\sin v+m} \qquad Y=\frac{-\cos u.v'+m\sin u\cos v}{\sin v+m}$$

$$Z=\frac{1+m\sin v}{\sin v+m} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (18)$$

As m and v are functions of u; X, Y, and Z in (18) represent the functions involving the paramater u, so they are considered the equations of the space curve in § 1.

Then by the method explained in § 1, we can determine an isometric coordinate system on the tangent surface of (18). In generally the parameter u is not the arc length of the curve (18), but by suitable transformation of variable, we can introduce the variable s on it, and determine the derivatives X_s , Y_s, and Z_s.

It is not so simple to solve the equation (17), the form of the equation of the tangent surface is not very practical, in generally. But when m takes some special functions of u, it is solved very simply, and the results are very practical.

§3. Solution of the Differential Equation $\sin v.v'^2+\cos v.v''=m(\cos^2 v+v'^2)$.

Transforming the equation to the form

$$\cos v\frac{d^2v}{du^2}+(\sin v-m)\left(\frac{dv}{du}\right)^2=m\cos^2 v\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (19)$$

it is evident that the independent variable u is not contained explicitly in it. Putting $\frac{dv}{du}$ to p, $\frac{d^2v}{du^2}$ is reduced to $p\frac{dp}{dv}$, so the equation (19) is deformed to

$$\frac{dp}{dv}+\left(\tan v-\frac{m}{\cos v}\right)p=\frac{m}{p}\cos v\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (20)$$

As the equation (20) is a Bernoulli's type, so it is solved to the form

$$p^2=2e^{-2\int\left(\tan v-\frac{m}{\cos v}\right)dv}\left\{m\int\cos v e^{2\int\left(\tan v-\frac{m}{\cos v}\right)dv}dv+c\right\}. \qquad \cdots\cdots (21)$$

From (21) v is determined as a function of u, by the relation $\frac{dv}{du}=p$. The expression of (21) is not so simple to determine v easily. Then we shall consider some special cases, in which v is determined simply.

[A] $m=0$

The equation (19) is reduced to

$$\cos v\frac{d^2v}{du^2}+\sin v\left(\frac{dv}{du}\right)^2=0\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (22)$$

and then (20) is reduced to

$$\frac{dp}{du}+\tan v\, p=0\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (23)$$

Then from (21) we have

$$p^2=2ce^{-2\int\tan v dv}, \quad \cdots\cdots\cdots\cdots (24) \quad \text{then} \quad p=ce^{-\int\tan v dv}. \quad \cdots\cdots\cdots\cdots (25)$$

However $\int \tan v dv = -\log \cos v$, so we have from (25),

$$\frac{dv}{du} = p = ce^{\log \cos v} = c \cos v . \qquad \cdots\cdots\cdots\cdots\cdots\cdots (26)$$

Integrating (26) we have

$$\log \tan\left(\frac{v}{2} + \frac{\pi}{4}\right) = cu . \qquad \cdots\cdots\cdots\cdots\cdots\cdots (27)$$

So in the case of $m=0$, the envelope of the family of planes (13) tangents to the globular surface along a loxodrome on it[2]. So we shall determine the equation ot edge of regression in these cases.

In (18), replacing $m=0$ and $v'=c \cos v$, we have

$$X = \frac{c \sin u \cos v}{\sin v} \qquad Y = \frac{-c \cos u \cos v}{\sin v} \qquad Z = \frac{1}{\sin v}, \qquad \cdots\cdots\cdots (28)$$

From (28), we have $\quad X_u = \dfrac{c\{\cos u \cos v \sin v - c \sin u \cos v\}}{\sin^2 v}$

$$Y_u = \frac{c\{\sin u \cos v \sin v + c \cos u \cos v\}}{\sin^2 v} \qquad Z_u = \frac{c \cos^2 v}{\sin^2 v}, \qquad \cdots\cdots (29)$$

and then $\quad X_u^2 + Y_u^2 + Z_u^2 = \dfrac{c^2 \cos^2 v (1+c^2)}{\sin^4 v}$.

Now transforming the independent variable u to the arc length s, we have

$$\sqrt{X_u^2 + Y_u^2 + Z_u^2}\,\frac{du}{ds} = \frac{c\sqrt{1+c^2}}{\sin^2 v} \cos v \frac{du}{ds} = 1$$

then we get $\quad \dfrac{du}{ds} = \dfrac{\sin^2 v}{c\sqrt{1+c^2} \cos v}$. $\qquad \cdots\cdots\cdots\cdots\cdots\cdots (30)$

From (30) the direction cosines of the tangent line at a given point on the edge of regression are given by the followings.

$$\alpha_x = \frac{dx}{ds} = \frac{dx}{du}\frac{du}{ds} = \frac{\cos u \sin v - c \sin u}{\sqrt{1+c^2}}$$

$$\alpha_y = \frac{\sin u \sin v + c \cos u}{\sqrt{1+c^2}} \qquad \alpha_z = \frac{\cos v}{\sqrt{1+c^2}}. \qquad \cdots\cdots\cdots\cdots\cdots (31)$$

From (31), differentiating αs with respect to s we have the direction cosines of the principal normal at the same point.

From the relations

$$\frac{d\alpha_x}{ds} = \frac{d\alpha_x}{du}\frac{du}{ds}, \qquad \sqrt{\sum\left(\frac{d\alpha_x}{ds}\right)^2} = \kappa \qquad \cdots\cdots\cdots\cdots\cdots\cdots (32)$$

and $\quad \dfrac{d\alpha_x}{ds} = \kappa\beta_x$, $\qquad \cdots\cdots\cdots\cdots\cdots\cdots (33)$

replacing the relation (30) to $\frac{du}{ds}$ in (32) and (33), we have

$$\kappa = \frac{\sin^3 v}{c\sqrt{1+c^2}\cos v} \qquad \text{and}$$

$$\beta_x = \frac{-1}{\sqrt{1+c^2}}(\sin u + c\cos u \sin v), \quad \beta_y = \frac{1}{\sqrt{1+c^2}}(\cos u - c\sin u \sin v)$$

$$\beta_z = \frac{-1}{\sqrt{1+c^2}}(c\cos v), \qquad \cdots\cdots\cdots\cdots (35)$$

where κ is the curvature of the edge of regression at the same point.

For these relations above, we can determine an isometric coordinate system on the envelope surface by the method explained in § 1.

[**B**] $m = \sin v$

The equation (20) is reducible to

$$\frac{dp}{dv} = \frac{\sin p \cos v}{p}. \qquad \cdots\cdots\cdots\cdots (36)$$

Integrating (36) we have

$$p^2 = 2\int \sin v \cos v dv + c = \sin^2 v + c$$

and then it follows that

$$p = \pm\sqrt{\sin^2 v + c}, \quad \frac{dv}{du} = \pm\sqrt{\sin^2 v + c} \quad \text{and} \quad du = \frac{dv}{\pm\sqrt{\sin^2 v + c}}. \quad \cdots (37)$$

Generally, the integral of (37) is reducible to an elliptic integral, excepting the case where $c = 0$; in the later case we have from (37) $\frac{dv}{du} = \pm \sin v$, so the integral is reducible to $\log\tan\frac{v}{2} = \pm u + c$.

So it is to say, that in the case where $c = 0$, the integral is reducible to the case explained in [A].

Conclusions

In this paper we have discussed only the case where the envelope surface tangents to the globular surface along a loxodrome on it, and did not refer to the cases, where m takes an arbitrary function. In the cases, where the globular surface is considered an ellipsoid of revolution, we refered nothing in the present paper, so we wish to discuss the leaved cases in the next chance.

Refferences

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