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ON GENERALIZED DEVELOPMENT PROJECTIONS

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Introduction

To get a method of map projections is to be regarded to construct a one to one corespondence between a point on the globular surface and a point on the given plane.

After all, various kinds of problems of map projections are reduced to the discussions of mapping functions. Under geometrical interpretations, there are two methods of map projections, the one is to project the surface to a developable surface, and then develope it on a plane; the other is to project to a plane directly. Under analytical considerations there is no distinction between these two methods, but by the former, we have oftenly more superior distortionless projections for simpler mapping functions; so the conical or cylindrical projections have been applied for a long time. In this paper we wish to consider the generalization of development projections. As their computations are not so simple, it is not always prospective to get some practical ones usually; but in the special cases, there are some expectations to get useful projections by these methods. Then we wish to explain for general theories on the generalized projections.

§ 1. Isometric Coordinate Systems on a Tangent Surface of a Given Curve.

Put the equations of a space curve to

$$x^i = x^i(s) \quad (i = 1, 2, 3) \qquad \cdots\cdots\cdots (1)$$

where the parameter s means the arc length of the curve measured from a given point.

Then the equations of a tangent line at a point on the curve are given by

$$X^i = x^i(s) + t\alpha^i(s) \,[1], \qquad \cdots\cdots\cdots (2)$$

where X^is mean the current coordinates and t is a parameter which means the distance from the tangent point to any point on the tangent line, and α^is mean the direction cosines of the tangent line.

If we consider s and t two independent variables in the equations (2), (2) represents the tangent surface of the curve (1). To obtain the fundamental magnitudes of (2), we consider the equations

$$\frac{d\alpha^i}{ds} = \kappa\beta^i , \quad \sum \alpha^i\beta^i = 0 , \tag{3}$$

where $\beta^i s$ mean the direction cosines of the principal normal of the curve (1), and κ means the curvature of the curve at the tangent point on it. Differentiating (2), with respect to s and t, we have

$$\frac{\partial X^i}{\partial s} = \alpha^i + t\kappa\beta^i , \quad \frac{\partial X^i}{\partial t} = \alpha^i ,$$

then we have

$$\sum \left(\frac{\partial X^i}{\partial s}\right)^2 = \sum \alpha^{i2} + 2t\kappa \sum \alpha^i\beta^i + t^2\kappa^2 \sum \beta^{i2} = 1 + t^2\kappa^2 ,$$

$$\sum \left(\frac{\partial X^i}{\partial t}\right)^2 = 1 , \quad \sum \left(\frac{\partial X^i}{\partial s}\right)\left(\frac{\partial X^i}{\partial t}\right) = 1 .$$

Then the line element of (2) is given by

$$ds^2 = dt^2 + 2dt\, ds + (1 + t^2\kappa^2)\, ds^2 , \quad \cdots\cdots\cdots\cdots\cdots\cdots \tag{4}$$

where κ is a function of s.

In the equations (4) t and s are not the parameters for isometric coordinate systems, so we wish to get an isometric coordinate system on it from the equation (4). Deforming (4), we have

$$ds^2 = \{dt + (1 + it\kappa)ds\}\{dt + (1 - it\kappa)ds\}$$

then we put

$$dt + (1 + it\kappa)ds \text{ to } pd\lambda , \quad \text{and} \quad dt + (1 - it\kappa)ds \text{ to } qd\mu . \quad \cdots\cdots \tag{5}$$

In (5) p and q mean the integrating factors of the expression, so we have from the integration of (5),

$$\lambda = \left(\frac{1}{\kappa}\tan^{-1} t\kappa + s\right) + \frac{i}{2\kappa}\log(1 + t^2\kappa^2)$$

$$\mu = \left(\frac{1}{\kappa}\tan^{-1} t\kappa + s\right) - \frac{i}{2\kappa}\log(1 + t^2\kappa^2)$$

Putting the real and imaginary parts of the above to x and y we have

$$x = \frac{1}{\kappa}\tan^{-1} t\kappa + s , \quad y = \frac{1}{2\kappa}\log(1 + t^2\kappa^2) . \quad \cdots\cdots\cdots\cdots\cdots \tag{6}$$

Then the equation (4) is reduced to

$$ds^2 = (1 + t^2\kappa^2)(dx^2 + dy^2) . \quad \cdots\cdots\cdots\cdots\cdots\cdots \tag{7}$$

In (7) parameters x and y are evidently isometric.

Then we have easily some conformal projections between the globular surface and the surface (2), applying the theories of functions of a complex variable. Developing the surface (2) to a plane, we have some new conformal projections of the earth.

§ 2. Envelope along a Given Curve on the Surface.

Put the equations of a given surface to

$$x = x(u\,v) \qquad y = (u\,v) \qquad z = z(u\,v)\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

then the equation of the tangent plane at the point $(x_0 y_0 z_0)$ on it is expressible to

$$\begin{vmatrix} X-x_0 & Y-y_0 & Z-z_0 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = 0\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (9)$$

where X, Y, Z, are current coordinates and x_u, y_v, etc. are partial derivatives of x and y with respect to u and v.

Specially, if (8) represent a surface of revolution, they are given by

$$x = p(v)\cos u \qquad y = p(v)\sin u \qquad z = q(v)\,, \qquad \cdots\cdots\cdots\cdots (10)$$

in (10) $p(v)$, $q(v)$ are arbitrary functions of v.

Then the equation (9) is reducible to

$$X\dot{q}\cos u + Y\dot{q}\sin u - Z\dot{p} = (p\dot{q} - \dot{p}q)\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (11)$$

Similarly, if the equations (8) represent a sphere, they are given by

$$x = \sin v \cos u \qquad y = \sin v \sin u \qquad z = \cos v\,, \qquad \cdots\cdots\cdots\cdots (12)$$

then (9) is reducible to

$$X\cos u \cos v + Y\sin u \cos v + Z\sin v = 1\,. \qquad \cdots\cdots\cdots\cdots (13)$$

If we consider v as a function of u in (11) and (13), they are equations of one parameter family of planes. Then these planes decide an envelope, of course, it is a developable surface.

If we can put the equation (11) or (13) to the form (2), it is able to determine an isometric coordinate system on it by the method explained in § 1.

To do so, we must determine the equation of edge of regression of the surface.

If the equation of family of planes involving a parametes t is given by

$$a(t)x+b(t)y+c(t)z+d(t)=0\,, \qquad \cdots\cdots\cdots\cdots (14)$$

the equation of edge of regression of the family is obtained by solving x, y, z as functions of t from (14) and the followings,

$$\dot{a}(t)x+\dot{b}(t)y+\dot{c}(t)z+\dot{d}(t)=0 \qquad \ddot{a}(t)x+\ddot{b}(t)y+\ddot{c}(t)z+\ddot{d}(t)=0\,,$$

where $\dot{a}(t)$, $\ddot{a}(t)$, etc. mean the first and second derivatives of the coefficients with respect to t.

In this paper, for convenience of computations, we shall start from the equation (13).

From (13), describing the equations corresponding to the above three equations, we have,

$$\begin{aligned}
&X\cos u \operatorname{cus} v+Y\sin u\cos v+Z\sin v=1\\
&X(\sin u\cos v+\cos u \operatorname{sins} v.v')+Y(-\cos u\cos v+\operatorname{sins} u \operatorname{sins} v.v')-Z\cos v.v'=0\\
&X(2\sin u \operatorname{sins} v.v'-\cos u \operatorname{sins} v.v'')+Y(-2\cos u \operatorname{sins} v.v'-\sin u \operatorname{sins} v.v'')\\
&\quad +Z(\cos v.v''+\operatorname{sins} v)=v'^2+1\,, \qquad \cdots\cdots\cdots\cdots (15)
\end{aligned}$$

where v', v'' mean the first and second derivatives of v with respect to u.

Solving (15) with respect to x, y, z, we have

$$\begin{aligned}
X&=\frac{\sin u.v'(\cos^2 v+v'^2)+\cos u\cos v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')}\\
Y&=\frac{-\cos u.v'(\cos^2 v+v'^2)+\sin u\cos v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')}\\
Z&=\frac{(\cos^2 v+v'^2)+\sin v(\sin v.v'^2+\cos v.v'')}{\sin v(\cos^2 v+v'^2)+(\sin v.v'^2+\cos v.v'')} \qquad \cdots\cdots\cdots\cdots (16)
\end{aligned}$$

As the equation (16) involving a parameter u, they represent the edge of regression of envelope of the family of planes (13).

Introducing m from the equation

$$\sin v.v'^2+\cos v.v''=m(\cos^2 v+v'^2)\,, \qquad \cdots\cdots\cdots\cdots (17)$$

the equations (16) are reducible to

$$\begin{aligned}
&X=\frac{\sin u.v'+m\cos u\cos v}{\sin v+m} \qquad Y=\frac{-\cos u.v'+m\sin u\cos v}{\sin v+m}\\
&Z=\frac{1+m\sin v}{\sin v+m} \qquad \cdots\cdots\cdots\cdots (18)
\end{aligned}$$

As m and v are functions of u ; X, Y, and Z in (18) represent the functions involving the paramater u, so they are considered the equations of the space curve in § 1.

Then by the method explained in § 1, we can determine an isometric coordinate system on the tangent surface of (18). In generally the parameter u is not the arc length of the curve (18), but by suitable transformation of variable, we can introduce the variable s on it, and determine the derivatives X_s , Y_s, and Z_s.

It is not so simple to solve the equation (17), the form of the equation of the tangent surface is not very practical, in generally. But when m takes some special functions of u, it is solved very simply, and the results are very practical.

§3. Solution of the Differential Equation $\sin v.v'^2+\cos v.v''=m(\cos^2 v+v'^2)$.

Transforming the equation to the form

$$\cos v\frac{d^2v}{du^2}+(\sin v-m)\left(\frac{dv}{du}\right)^2=m\cos^2 v\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (19)$$

it is evident that the independent variable u is not contained explicitly in it. Putting $\frac{dv}{du}$ to p, $\frac{d^2v}{du^2}$ is reduced to $p\frac{dp}{dv}$, so the equation (19) is deformed to

$$\frac{dp}{dv}+\left(\tan v-\frac{m}{\cos v}\right)p=\frac{m}{p}\cos v\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (20)$$

As the equation (20) is a Bernoulli's type, so it is solved to the form

$$p^2=2e^{-2\int\left(\tan v-\frac{m}{\cos v}\right)dv}\left\{m\int\cos v e^{2\int\left(\tan v-\frac{m}{\cos v}\right)dv}dv+c\right\}. \quad \cdots\cdots (21)$$

From (21) v is determined as a function of u, by the relation $\frac{dv}{du}=p$. The expression of (21) is not so simple to determine v easily. Then we shall consider some special cases, in which v is determined simply.

[A] $m=0$

The equation (19) is reduced to

$$\cos v\frac{d^2v}{du^2}+\sin v\left(\frac{dv}{du}\right)^2=0\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (22)$$

and then (20) is reduced to

$$\frac{dp}{du}+\tan v\, p=0\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (23)$$

Then from (21) we have

$$p^2=2ce^{-2\int\tan v dv}, \quad \cdots\cdots\cdots\cdots (24) \quad \text{then} \quad p=ce^{-\int\tan v dv}. \quad \cdots\cdots\cdots\cdots (25)$$

However $\int \tan v dv = -\log\cos v$, so we have from (25),

$$\frac{dv}{du} = p = ce^{\log\cos v} = c\cos v\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (26)$$

Integrating (26) we have

$$\log\tan\left(\frac{v}{2}+\frac{\pi}{4}\right) = cu\,. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (27)$$

So in the case of $m=0$, the envelope of the family of planes (13) tangents to the globular surface along a loxodrome on it[2]. So we shall determine the equation ot edge of regression in these cases.

In (18), replacing $m=0$ and $v'=c\cos v$, we have

$$X = \frac{c\sin u\cos v}{\sin v} \qquad Y = \frac{-c\cos u\cos v}{\sin v} \qquad Z = \frac{1}{\sin v}, \qquad \cdots\cdots\cdots (28)$$

From (28), we have $\quad X_u = \dfrac{c\{\cos u\cos v\sin v - c\sin u\cos v\}}{\sin^2 v}$

$$Y_u = \frac{c\{\sin u\cos v\sin v + c\cos u\cos v\}}{\sin^2 v} \qquad Z_u = \frac{c\cos^2 v}{\sin^2 v}, \qquad \cdots\cdots (29)$$

and then $\quad X_u^2 + Y_u^2 + Z_u^2 = \dfrac{c^2\cos^2 v(1+c^2)}{\sin^4 v}.$

Now transforming the independent variable u to the arc length s, we have

$$\sqrt{X_u^2+Y_u^2+Z_u^2}\,\frac{du}{ds} = \frac{c\sqrt{1+c^2}}{\sin^2 v}\cos v\,\frac{du}{ds} = 1$$

then we get $\quad \dfrac{du}{ds} = \dfrac{\sin^2 v}{c\sqrt{1+c^2}\cos v}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (30)$

From (30) the direction cosines of the tangent line at a given point on the edge of regression are given by the followings.

$$\alpha_x = \frac{dx}{ds} = \frac{dx}{du}\frac{du}{ds} = \frac{\cos u\sin v - c\sin u}{\sqrt{1+c^2}}$$

$$\alpha_y = \frac{\sin u\sin v + c\cos u}{\sqrt{1+c^2}} \qquad \alpha_z = \frac{\cos v}{\sqrt{1+c^2}}. \qquad \cdots\cdots\cdots\cdots\cdots (31)$$

From (31), differentiating αs with respect to s we have the direction cosines of the principal normal at the same point.

From the relations

$$\frac{d\alpha_x}{ds} = \frac{d\alpha_x}{du}\frac{du}{ds}, \qquad \sqrt{\sum\left(\frac{d\alpha_x}{ds}\right)^2} = \kappa \qquad \cdots\cdots\cdots\cdots\cdots\cdots (32)$$

and $\quad \dfrac{d\alpha_x}{ds} = \kappa\beta_x, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (33)$

replacing the relation (30) to $\frac{du}{ds}$ in (32) and (33), we have

$$\kappa = \frac{\sin^3 v}{c\sqrt{1+c^2}\cos v} \qquad \text{and}$$

$$\beta_x = \frac{-1}{\sqrt{1+c^2}}(\sin u + c\cos u \sin v), \quad \beta_y = \frac{1}{\sqrt{1+c^2}}(\cos u - c\sin u \sin v)$$

$$\beta_z = \frac{-1}{\sqrt{1+c^2}}(c\cos v), \qquad \cdots\cdots (35)$$

where κ is the curvature of the edge of regression at the same point.

For these relations above, we can determine an isometric coordinate system on the envelope surface by the method explained in § 1.

[**B**] $m = \sin v$

The equation (20) is reducible to

$$\frac{dp}{dv} = \frac{\sin p \cos v}{p}. \qquad \cdots\cdots (36)$$

Integrating (36) we have

$$p^2 = 2\int \sin v \cos v dv + c = \sin^2 v + c$$

and then it follows that

$$p = \pm\sqrt{\sin^2 v + c}, \quad \frac{dv}{du} = \pm\sqrt{\sin^2 v + c} \quad \text{and} \quad du = \frac{dv}{\pm\sqrt{\sin^2 v + c}}. \quad \cdots (37)$$

Generally, the integral of (37) is reducible to an elliptic integral, excepting the case where $c = 0$; in the later case we have from (37) $\frac{dv}{du} = \pm\sin v$, so the integral is reducible to $\log\tan\frac{v}{2} = \pm u + c$.

So it is to say, that in the case where $c = 0$, the integral is reducible to the case explained in [A].

Conclusions

In this paper we have discussed only the case where the envelope surface tangents to the globular surface along a loxodrome on it, and did not refer to the cases, where m takes an arbitrary function. In the cases, where the globular surface is considered an ellipsoid of revolution, we refered nothing in the present paper, so we wish to discuss the leaved cases in the next chance.

Refferences

1) Page 38. An introduction to differential geometry; Eisenhart, Princeton University Press 1947.

2) Page 72. Mathematische Grundlagen der Höheren Geodesie und Kartographie; König, Springer 1951.