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ON THE MODIFIED COSINE FUNCTIONS

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In the previous note,[1] we had found, as a particular solution of the differential equation

$$\frac{d^2y}{dz^2}-\frac{2n}{z}\frac{dy}{dz}+\left(1+\frac{2n}{z^2}\right)y=0, \tag{1}$$

an integral transcendental function

$$V_n(z)=\left[\frac{d^n}{d\zeta^n}(\cos\sqrt{1+2\zeta}\,z)\right]_{\zeta=0}=(-2z^2)^n\sum_{m=0}^{\infty}\frac{(-1)^m\,\underline{|m+n}}{\underline{|m}\,\underline{|2(m+n)}}z^{2m},$$
$$(n=0,1,2,\cdots). \tag{2}$$

In the present note, we shall discuss the general solution of the differential equation, rather more generalized than (1), and the properties of the functions $V_n(z)$, which appear very similar to those of Bessel functions $J_n(z)$.

Although the differential equation (1) can be classified into the Bessel's equation in a broader sense,[2] yet its form is surely different from the ordinary Bessel equation $\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\left(1-\frac{n^2}{z^2}\right)y=0$. So we venture to treat it below.

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[1] This Journal, Vol. IV (1954), p. 39, Y. Watanabe and M. Nakamura, On the Partial Differential Equation of Paraboric Type with Constant Coefficients.

[2] *Cf. e. g.* Whittaker and Watson, Modern Analysis, 3rd Edition, p. 203–4. Namely, the diffierential equation of the form

$$\frac{d^2u}{d\zeta^2}+\sum_{r=1}^{4}\frac{\frac{1}{2}-2\alpha_r}{\zeta-a_r}\frac{du}{d\zeta}+\left\{\sum_{r=1}^{4}\frac{\alpha_r\left(\alpha_r+\frac{1}{2}\right)}{(\zeta-a_r)^2}+\frac{A\zeta^2+2B\zeta+C}{\prod_{r=1}^{4}(\zeta-a_r)}\right\}u=0,$$

where $A=(\sum_{r=1}^{4}\alpha_r)^2-\sum_{r=1}^{4}\alpha_r^2+\frac{3}{2}\sum_{r=1}^{4}\alpha_r+\frac{3}{16}$ and B, C are constants, is called the generalized Lamé's equation. This differential equation has every point in the whole ζ-plane, except a_1, a_2, a_3, a_4, and ∞, as an ordinary point, these five points being all regular points with exponents α_r, $\alpha_r+\frac{1}{2}$ at a_r $(r=1, 2, 3, 4)$ and μ, $\mu+\frac{1}{2}$ at ∞. If we make two or more of these five singular points to tend to coincidence, we obtain thereby the so-called confluent equations. Among them, there is such a type which has only one regular, and only one irregular singularity, and else everywhere as ordinary behaves, and its type is called the (generalized) Bessel's equation. In this broader definition, no doubt, our present modified cosine function belongs to the (generalized) Bessel's functions. Therefore it will be more preferable to discuss more generally the Bessel equation in this broader sence. However we reserve this problem as a further task.

§ 1. We consider the differential equation of the form (1), but now n being not necessarily confined as a positive integer:

$$z^2\frac{d^2y}{dz^2}-2nz\frac{dy}{dz}+(z^2+2n)y=0,$$

whose singularity occurs regularly at $z=0$ but irregularly at $z=\infty$[3]. Let us find its formal solution

$$y=\sum_{\nu=0}^{\infty}a_\nu z^{\alpha+\nu},$$

where the index α and coefficients a_ν's are to be determined. Substituting in the differential equation, we get

$$\sum_{\nu=0}^{\infty}\Big[\{(\nu+\alpha)(\nu+\alpha-1)-2n(\nu+\alpha)+2n\}\,a_\nu+a_{\nu-2}\Big]z^{\alpha+\nu}=0.$$

Equating coefficients of successive powers of z to zero, we obtain

$$(\nu+\alpha-2n)(\nu+\alpha-1)\,a_\nu+a_{\nu-2}=0 \qquad \begin{pmatrix}\nu=0,1,2,\cdots \\ a_{-1}=0, \quad a_{-2}=0\end{pmatrix}. \qquad (3)$$

So for $\nu=0$

$$(\alpha-2n)(\alpha-1)\,a_0=0.$$

Hence the indicial equation has the roots $\alpha=2n$ and 1.

Firstly, taking $\alpha=2n$, we obtain the recurring formula

$$\nu(\nu+2n-1)\,a_\nu+a_{\nu-2}=0, \quad i.e, \qquad a_\nu=\frac{-a_{\nu-2}}{\nu(\nu+2n-1)}.$$

Hence, except $2n-1=$ *negative even*, $-2q$ say, we get for $\nu=2m$

$$a_{2m}=\frac{-a_{2m-2}}{2m(2m+2n-1)}=\frac{a_{2m-4}}{2^2m(m-1)(2m+2n-1)(2m+2n-3)}$$
$$=\cdots=\frac{(-1)^m\,\Gamma(2n+1)\,\Gamma(m+n+1)}{\underline{|m}\,\Gamma(n+1)\,\Gamma(2m+2n+1)}\,a_0 \quad \begin{pmatrix}m=1,2,\cdots \\ n\neq -q+\frac{1}{2}\end{pmatrix},$$

which on putting $a_0=\dfrac{\Gamma(n+1)}{\Gamma(2n+1)}(-2)^n$ reduces to

$$a_{2m}=\frac{(-1)^{m+n}\,\Gamma(m+n+1)\,2^n}{\underline{|m}\,\Gamma(2m+2n+1)}.$$

For the sake of convenience[4] we may assume all $a_{2m+1}=0$ $(m=0,1,\cdots)$,

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3) Writing $z=\dfrac{1}{z_1}$, the equation (1) becomes $\dfrac{d^2y}{dz_1^2}+\dfrac{2(n+1)}{z_1}\dfrac{dy}{dz_1}+\dfrac{1+2nz_1^2}{z_1^4}y=0$, and thus the coefficient of y has a pole of order 4 at $z_1=0$, *i. e.* at $z=\infty$, hence there the equation is irregular (Unbestimmtheitsstelle).

4) Moreover, if n is neither negative integer nor 0, it is necessarily all $a_{2m+1}=0$, because, then we should have by (3) $2na_1=0$, as well as $(2m+1)\,2(m+n)\,a_{2m+1}+a_{2m-1}=0$, $(m=1,2,3,\cdots)$.

since they are quite independent of a_{2m} and surely satisfy the relation (3). Thus in case $2n-1 \neq -2q$ (even negative), we obtain, as the first particular solution, an infinite series $V_n(z)$:

$$y_1 = V_n(z) = (-2z^2)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|m}} \frac{\Gamma(m+n+1)}{\Gamma(2m+2n+1)} z^{2m}, \tag{4}$$

which is equal to $\left[\frac{d^n}{d\zeta^n}(\cos\sqrt{1+2\zeta}\, z)\right]_{\zeta=0}$, only when n is a positive integer or zero[5].

Next, taking the second root $\alpha=1$, we obtain another recurring formula

$$\nu(\nu-2n+1)\, a_\nu + a_{\nu-2} = 0\,,$$

which can be availed for even ν if $2n-1$ be not positive even. Thus we get

$$a_{2m} = \frac{-a_{2m-2}}{2m(2m-2n+1)} = \cdots = \frac{(-1)^m a_0}{2^m \underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)}\,.$$

Hence, on putting again all $a_{2m+1}=0$, we obtain, as the second particular solution,

$$y_2 = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{2^m \underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)} \qquad \left(n \neq q+\frac{1}{2}\right). \tag{5}$$

But if n is a positive integer, we have

$$\begin{aligned}\left[\frac{d^n}{d\zeta^n}\sin(\sqrt{1+2\zeta}\, z)\right]_{\zeta=0} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} z^{2m+1} \left[\frac{d^n}{d\zeta^n} \sum_{l=0}^{\infty} \binom{m+\frac{1}{2}}{l} 2^l \zeta^l\right]_{\zeta=0} \\ &= \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} z^{2m+1} \sum_{l=n}^{\infty} \frac{\Gamma\left(m+\frac{3}{2}\right) 2^l}{\Gamma\left(m-l+\frac{3}{2}\right)} \frac{\zeta^{l-n}}{\underline{|l-n}}\right]_{\zeta=0} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m+1}} (2m+1)(2m-1)\cdots(2m-2n+3)\, z^{2m+1}. \end{aligned} \tag{6}$$

Now, in order to equalize (5) and (6), we put the reserved constant

$$a_0 = \frac{(-1)^{n-1}\, \underline{|2n-2}}{2^{n-1}\, \underline{|n-1}} \qquad (n \geqq 1)\,, \tag{7}$$

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5) Even though we take the Riemann-Liouville's fractional derivative, *e. g.* of order $n=1-\alpha$, $0<\alpha<1$, formally we get

$$\begin{aligned} D^n \cos\sqrt{1+2\zeta} z = DI^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{|2m}} (1+2\zeta)^m &= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{|2m}} \sum_{l=0}^{m} \frac{\underline{|m}\, 2^l}{\underline{|m-l}} DI^\alpha \frac{\zeta^l}{\Gamma(l+1)} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\underline{|2m}} \sum_{l=0}^{m} \frac{\underline{|m}}{\underline{|m-l}} \frac{2^l \zeta^{l+\alpha-1}}{\Gamma(l+\alpha)}\,. \end{aligned}$$

To put here $\zeta=0$, it is no more than to obtain an absurd result

$$\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{\underline{|2m}} z^{2m} \frac{\zeta^{\alpha-1}}{\Gamma(\alpha)}\right]_{\zeta\to 0} = \text{indeterminate}.$$

then the expression (5) just coincides with (6). Really, by division of the corresponding summands, we get

$$\frac{a_0(-1)^m}{2^m\,\underline{|m}(2m-2n+1)\cdots(-2n+3)}\div\frac{(-1)^m}{\underline{|2m+1}}(2m+1)(2m-1)\cdots(2m-2n+3)$$
$$=\frac{a_0\quad \underline{|2m+1}}{2\cdot4\cdots2m(2m+1)(2m-1)\cdots(2m-2n+3)(2m-2n+1)\cdots3\cdot1\cdot(-1)(-3)\cdots(-2n+3)}$$
$$=\frac{(-1)^{n-1}\,\underline{|2n-2}}{2^{n-1}\,\underline{|n-1}}\;\frac{2^{n-1}\,\underline{|n-1}}{(-1)^{n-1}\,\underline{|2n-2}}=1\,.$$

It is noteworthy to observe that the number of the linear factors in the denominator of summand in (5) is just m, so that it is available irrespectively whether n is a positive integer or not. However, in (6), the number of the linear factors in the numerator of summand is exactly n, and consequently (6) is not legitimate unless n is a positive integer. Hence, in general, we adopt the former and put

$$U_n(z)=\left(\frac{-1}{2}\right)^{n-1}\frac{\Gamma(2n-1)}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{(-1)^m z^{2m+1}}{2^m\,\underline{|m}(2m-2n+1)(2m-2n-1)\cdots(-2n+3)} \tag{8}$$

which gives another particular solution, if $n-\frac{1}{2}\neq q$ (positive integer).

Thus, for any real n, except some trivial cases, we have obtained, as two particular solutions $V_n(z)$ and $U_n(z)$, generally independent of each other. In particular, when n is a positive integer, the two series $V_n(z)$ and $U_n(z)$ becomes

$$\begin{matrix}V_n(z)\\U_n(z)\end{matrix}=\left[\frac{d^n}{d\zeta^n}\begin{matrix}\cos\\\sin\end{matrix}(\sqrt{1+2\zeta}\,z)\right]_{\zeta=0}, \tag{9}$$

and might be called *modified cosine-* and *modified sine- functions* respectively. Surely they are independent of each other, as one is even function while the other is odd. Generally the general solution of (1) is given by

$$y=AV_n(z)+BU_n(z)\,, \tag{10}$$

where A and B are arbitrary constants. Of course, to say more exactly we have to examine several exceptional cases more minutely, and to secure valid solutions. The exceptional cases may occur when the difference of exponents $2n-1$ becomes an integer or zero. *E.g.* in the latter case we have $n=\frac{1}{2}$, and our series then become coincident:

$$U_{\frac{1}{2}}(z)=V_{\frac{1}{2}}(z)=\sum_{m=0}^{\infty}\frac{(-1)^{m+\frac{1}{2}}\,\Gamma\left(m+\frac{3}{2}\right)}{\underline{|m}\,\Gamma(2m+2)}2^{\frac{1}{2}}z^{2m+1}$$
$$=C\sum_{m=0}^{\infty}\frac{(-1)^m}{(\underline{|m})^2}\left(\frac{z}{2}\right)^{2m+1}=CzJ_0(z)\,,$$

thus it reduces to the Bessel function of order 0 multiplied with z. However, we reserve the discussion of all such special cases for future, and presently mainly confining to the case that n is a positive integer, and also rather laying stress upon $V_n(z)$, we proceed to deduce their properties.

§ 2. In the previous note[6], it was seen that the relations

$$\frac{V_n(z)}{z^2}-\frac{V_n'(z)}{z}=V_{n-1}(z)\,, \quad i.e. \quad \left(\frac{1}{z}V_n(z)\right)'=-V_{n-1}(z) \tag{11}$$

and

$$V_n''(z)+V_n(z)=-2nV_{n-1}(z) \tag{12}$$

hold.

From (11) and (12) immediately follows

$$V_{n+1}(z)+(2n-1)\;V_n(z)+z^2V_{n-1}(z)=0\,, \tag{13}$$

and further this combined with (11) gives

$$V_n'(z)=\frac{2nV_n(z)}{z}+\frac{V_{n+1}(z)}{z}\,,$$

and also

$$\frac{d}{dz}\left(\frac{V_n(z)}{z^{2n}}\right)=\frac{V_{n+1}(z)}{z^{2n+1}}\,. \tag{14}$$

Since the form as the infinite series is invariable, whatever n may be, integral or fractional, all the above identities still hold even for non-positive integral n, so far as they exist.

Also if $\quad n>0\,,\quad \lim\limits_{z\to 0} V_n(z)=o(1)=o(z^{2n-\varepsilon}) \qquad (\varepsilon>0)$

and if $\quad n>\frac{1}{2}\,,\quad \lim\limits_{z\to 0} V_n'(z)=o(1)=o(z^{2n-1-\varepsilon})$.

§ 3. Now we shall prove the theorem that $V_n(z)$ with $n\geqslant 0$ has infinitely many real roots and moreover between any two consecutive real zeros of $V_n(z)$, there lies one and only one zero of $V_{n+1}(z)$. Since $V_n(z)$ is an even function, its real zero-points, if any besides $z=0$, should occur in pair of opposite signs with equal absolute value, so that we may only conceive its positive roots.

We prove the theorem by mathematical induction. At first for $V_0(z)=\cos z$ and $V_1(z)=-z\sin z$ the theorem is evident. Next let us assume that $V_n(z)=0$ and hence $V_n(z)/z^{2n}=0$ has infinitely many (discrete) roots,

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[6] loc. cip. p. 41.

In view of (14) together with Rolle's theorem, it follows that between each consecutive pair of zeros of $V_n(z)/z^{2n}$ there is at least one zero of $V_{n+1}(z)/z^{2n+1}$ Similarly, from (11) it follows that between each consecutive pair of zeros of $V_{n+1}(z)$ and hence of $V_{n+1}(z)/z$, there is at least one zero of $V_n(z)$. Therefore the theorem is true for $V_{n+1}(z)$, if it is true for $V_n(z)$. Hence it holds in general.

§ 4. To give another proof of the preceding theorem, we ready[7] an integral representation of $V_n(x)$: When $0<n<1$, $x=\frac{\pi\theta}{2}$, it holds that

$$V_n\left(\frac{\pi}{2}\theta\right)=\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)\pi^{2n-\frac{1}{2}}}{\Gamma(2n)\,2^n}\theta\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt\,. \tag{15}$$

To prove this, let us transform the integral

$$I=\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt$$

by putting $t=\theta\sqrt{s}$, as follows.

$$I=\int_0^1\frac{1}{\theta^{2(1-n)}(1-s)^{1-n}}\sum_{m=0}^{\infty}\frac{(-1)^m}{\underline{|2m}}\left(\frac{\pi}{2}\theta\right)^{2m}s^m\frac{\theta ds}{2\sqrt{s}}$$

$$=\frac{1}{2}\theta^{2n-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{\underline{|2m}}\left(\frac{\pi}{2}\theta\right)^{2m}\int_0^1(1-s)^{n-1}s^{m-\frac{1}{2}}ds$$

But, as

$$B\left(n,m+\frac{1}{2}\right)=\frac{(2m-1)(2m-3)\cdots3\cdot\sqrt{\pi}}{(2n+2m-1)(2n+2m-3)\cdots(2n+1)}\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)},$$

so becomes

$$I=\frac{\Gamma(2n+1)\,2^{n-1}\sqrt{\pi}}{n\theta\Gamma\left(n+\frac{1}{2}\right)\pi^{2n}}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+n+1)\,2^n}{\underline{|m}\,\Gamma(2m+2n+1)}\left(\frac{\pi}{2}\theta\right)^{2m+2n}$$

$$=\frac{(-1)^n2^n\Gamma(2n)\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)\theta\pi^{2n}}V_n\left(\frac{\pi}{2}\theta\right)=\frac{V_n\left(\frac{\pi}{2}\theta\right)}{C(n,\theta)},$$

whence (15) is proved.

As the coefficient $C(n,\theta)$ does not vanish in $0<\theta<\infty$, the vanishing of $V_n\left(\frac{\pi}{2}\theta\right)$ and that of the integral take place at the same time. Hence we have only to consider the change of the sign of the integral:

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[7] Y. Watanabe, Über die Verschiebung der Nullstellen usw., this Journal, vol. III (1953), p. 16.

$$sgn\ V_n\left(\frac{\pi}{2}\theta\right)=sgn\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt\,.$$

Divide the whole integration interval $0<t<\infty$ at the points $t=0,1,2,\cdots$. The function $\frac{1}{(\theta^2-t^2)^{1-n}}$ being monotonously increasing, if we put for any positive integer q

$$\int_{4q-2}^{4q}=v_{2q}\,,\quad\int_{4q}^{4q+2}=-v_{2q+1}\,,$$

it is clear that all v_p are positive and moreover v_p increases with p. Hence, if we write

$$\int_0^\theta\frac{\cos\frac{\pi}{2}t}{(\theta^2-t^2)^{1-n}}dt=\int_0^{2p+\alpha}=\int_0^2+\int_2^4+\cdots+\int_{2p-2}^{2p}+\int_{2p}^{2p+\alpha}$$
$$=-v_1+v_2-\cdots+(-1)^p v_p+(-1)^p v_p'\,,$$

then

$$sgn\ V_n\left(\pi\left(p+\frac{\alpha}{2}\right)\right)=sgn\left[-v_1+v_2-v_3+\cdots+(-1)^p v_p+(-1)^p v_p'\right],$$

where $\quad 0<v_1<v_2<\cdots<v_p\,,\;$ and also $v_p'\geqslant 0$ if $0\leqslant\alpha\leqslant 1$.

Therefore, according as $\quad p=even=2q\;$ or $\;p=odd=2q+1\,,$

$$sgn\ V_n\left(\pi\left(2q+\frac{\alpha}{2}\right)\right)=sgn\left[v_{2q}'+(v_{2q}-v_{2q-1})+\cdots+(v_2-v_1)\right]=+\,,$$
$$sgn\ V_n\left(\pi\left(2q+1+\frac{\alpha}{2}\right)\right)=sgn\left[-v_{2q+1}'-(v_{2q+1}-v_{2q})-\cdots-(v_3-v_2)-v_1\right]=-\,.$$

Thus the change of sign of $V_n(x)$ in $0<x<\infty$ happens an infinitely many times. The result just proved is obtained for the case $0<n<1$. However, it can be proved for the case $1<n<2$, $2<n<3,\cdots$, in the same way as done in § 3.

§ 5. Now we shall prove an integral theorem, which resembles to that of Lommel in regard to Bessel function. Let α and β be some different parameters $\neq 0$. Writing $z=\alpha x$ in (1), we have

$$\frac{d^2y}{dx^2}-\frac{2n}{x}\frac{dy}{dx}+\left(\alpha^2+\frac{2n}{x^2}\right)y=0\,,$$

one solution of which is obviously $V_n(\alpha x)$ and consequently

$$\frac{d^2V_n(\alpha x)}{dx^2}-\frac{2n}{x}\frac{dV_n(\alpha x)}{dx}+\left(\alpha^2+\frac{2n}{x^2}\right)V_n(\alpha x)=0\,.$$

Similarly

$$\frac{d^2V_n(\beta x)}{dx^2}-\frac{2n}{x}\frac{dV_n(\beta x)}{dx}+\left(\beta^2+\frac{2n}{x^2}\right)V_n(\beta x)=0\,.$$

Multiplying the former by $V_n(\beta x)$ and the latter by $V_n(\alpha x)$ respectively, and then subtructing side by side, we get

$$\frac{du}{dx}-\frac{2n}{x}u+(\alpha^2-\beta^2)\,V_n(\alpha x)V_n(\beta x)=0\,,$$

where

$$u=V_n(\beta x)\frac{dV_n(\alpha x)}{dx}-V_n(\alpha x)\frac{dV_n(\beta x)}{dx}\,.$$

Multiplying the differential equation just obtained by x^{-2n}, and integrating, it yields

$$\left[\frac{u}{x^{2n}}\right]_0^1=-(\alpha^2-\beta^2)\int_0^1\frac{1}{x^{2n}}V_n(\alpha x)V_n(\beta x)\,dx\,.$$

Since $\lim\dfrac{V_n(z)}{z^{2n}}=\mathit{finite}$, and $\dfrac{dV_n(z)}{dz}$ vanishes at $z=0$ for $n>\dfrac{1}{2}$, so also $\dfrac{u}{x^{2n}}$ vanishes at $x=0$ (and this is also true for $n=0$, because of $V_0'(z)=-\sin z$). Thus the integrated part reduces to

$$\left[\frac{u}{x^{2n}}\right]_0^1=V_n(\beta)V_n'(\alpha)-V_n(\alpha)V_n'(\beta)\,.$$

Consequently we have (at least, when $n=$ *positive integer* or 0)

$$\int_0^1\frac{V_n(\alpha x)V_n(\beta x)}{x^{2n}}dx=\frac{1}{\alpha^2-\beta^2}\{V_n(\alpha)V_n'(\beta)-V_n(\beta)V_n'(\alpha)\}\qquad(\alpha\neq\beta)\,.\quad(16)$$

If we make β tend to α, the right-handed side of (16) becomes an indeterminate form $\dfrac{0}{0}$. However, on using l'Hospital's rule, and referring to (1) and (11), we can easily find the limiting value to be

$$\int_0^1\frac{V_n(\alpha x)^2}{x^{2n}}dx=\frac{1}{2\alpha}\{V_n'(\alpha)^2-V_n(\alpha)V_n''(\alpha)\}=\frac{1}{2\alpha}\{V_n'(\alpha)^2+V_n(\alpha)^2+2nV_n(\alpha)V_{n-1}(\alpha)\}$$

$$=\frac{1}{2\alpha}\left\{\left(1+\frac{1}{\alpha^2}\right)V_n(\alpha)^2+\alpha^2V_{n-1}(\alpha)^2+2(n-1)V_n(\alpha)V_{n-1}(\alpha)\right\}.\qquad(17)$$

§ 6. By use of the foregoing theorem, we can prove that $V_n(z)=0$ has really real roots only. For, in the integral theorem (16), *i.e.*

$$(\alpha^2-\beta^2)\int_0^1\frac{V_n(\alpha x)V_n(\beta x)}{x^{2n}}dx=V_n(\alpha)V_n'(\beta)-V_n(\beta)V_n'(\alpha)\,,$$

let $\alpha=\xi+i\eta$ be any roots of $V_n(z)=0$, then $\beta=\xi-i\eta$ should be so also, because the expansion (2) of $V_n(z)$ has only real coefficients. Accordingly

$$\left[(\xi+i\eta)^2-(\xi-i\eta)^2\right]\int_0^1\frac{V_n((\xi+i\eta)x)V_n((\xi-i\eta)x)}{x^{2n}}dx=0\,.$$

Here the integrand has the form

$$\frac{1}{x^{2n}}(P+iQ)(P-iQ)=\frac{P^2+Q^2}{x^{2n}}\geqslant 0\,,$$

and P^2+Q^2 cannot be 0 throughout any subinterval. Therefore the above integral is surely positive. Hence the multiplied factor $4\xi\eta i$ must vanish, so that $\xi=0$ or $\eta=0$, *i. e.* the root must be pure imaginary or pure real. But it is evident that

$$V_n(\pm\eta i)=(-1)^n(\eta i)^{2n}\sum_{m=0}^{\infty}\frac{(-1)^m\,|\underline{m+n}}{|\underline{m}\,|\underline{2(m+n)}}\,2^n(\eta i)^{2m}$$

$$=\eta^{2n}\sum_{m=0}^{\infty}\frac{|\underline{m+n}}{|\underline{m}\,|\underline{2(m+n)}}\,2^n\eta^{2m}>0\,,$$

Hence there is no pure imaginary root. Therefore any possible roots should be purely real, and really they exist as already shown in §§ 3, 4.

§ 7. We shall expand an arbitrary function $f(x)$, which is $L(0,1)$, in a series of $V_n(z)$ in the form

$$f(x)=\sum_{r=1}^{\infty}A_r V_n(\lambda_r x)^{8)}\,, \tag{18}$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_r<\cdots$ denote real positive roots of $V_n(z)=0$. To determine a coefficient A_s, we multiply both members of (18) by $V_n(\lambda_s x)/x^{2n}$ and integrate from $x=0$ to $x=1$. Then by virtue of (16) and (17) we obtain

$$\int_0^1\frac{V_n(\lambda_s x)f(x)}{x^{2n}}\,dx=\int_0^1\sum_r=\sum_r A_r\int_0^1\frac{1}{x^{2n}}V_n(\lambda_r x)\,V_n(\lambda_s x)\,dx$$

$$=\frac{1}{2}A_s\lambda_s\{V_{n-1}(\lambda_s)\}^2\,.$$

Hence

$$A_s=2\int_0^1\frac{1}{x^{2n}}V_n(\lambda_s x)f(x)\,dx\Big/\lambda_s\{V_{n-1}(\lambda_s)\}^2\,,\qquad (s=1,2,3,\cdots)\,. \tag{19}$$

For instance, if $n=1$, we get

$$A_s=-2\int_0^1\frac{1}{x^2}\,s\pi x\sin(s\pi x)\cdot f(x)\,dx\Big/s\pi(\cos s\pi)^2$$

$$=-2\int_0^1\frac{1}{x}\sin(s\pi x)f(x)\,dx\,.$$

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8) We have tacitly assumed that $f(x)$ is coutinuous throughout the interval $(0,1)$. It can be proved more rigorously in just the same manner as shown by Hobson (Proc. London Math. Soc. 2, vol. VII, 1909, p.p. 387-8, or Watson, Theory of Bessel functions 1922, p. 591.), that if $f(x)$ is absolutely integrable and of bounded variation in $(0,1)$, then the series is convergent and its sum is $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

§ 8. Lastly we shall prove that the two functions $V_n(z)$ and $U_n(z)$ are connected by the relation

$$U_n(z)V_{n-1}(z)-U_{n-1}(z)V_n(z)=z^{2n-1} \qquad (n \geqslant 1). \tag{20}$$

For, since $U_n(z)$ and $V_n(z)$ satisfy the differential equation (1), we have

$$z^2U_n''(z)-2nzU_n'(z)+(z^2+2n)\,U_n(z)=0,$$
$$z^2V_n''(z)-2nzV_n'(z)+(z^2+2n)\,V_n(z)=0,$$

and whence

$$z^2(U_n''(z)V_n(z)-U_n(z)V_n''(z))-2nz(U_n'(z)V_n(z)-U_n(z)V_n'(z))=0,$$

that is

$$\frac{d}{dz}(U_n'(z)V_n(z)-U_n(z)V_n'(z))=\frac{2n}{z}(U_n'(z)V_n(z)-U_n(z)V_n'(z)).$$

On integrating we get

$$U_n'(z)V_n(z)-U_n(z)V_n'(z)=Cz^{2n}.$$

Substituting in the left handed side the value (11) of $V_n'(z)$ and similar one about $U_n'(z)$, which can be easily shown from the expansion (6) or (8), we get

$$U_n(z)V_{n-1}(z)-U_{n-1}(z)V_n(z)=Cz^{2n-1}.$$

To determine the value of C, we have only to find

$$\lim_{z\to 0}\left[\frac{U_n(z)}{z}\frac{V_{n-1}(z)}{z^{2(n-1)}}-U_{n-1}(z)\frac{V_n(z)}{z^{2n-1}}\right];$$

But it is easy to see that the limiting value becomes 1, by means of (4) (6) and (8), which proves (20).