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BIMODAL DISTRIBUTIONS[1]

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These are practically frequently of use. It is said that K. Pearson answered it by the superposition of his fundamental unimodal specimen curves. However, as I could neither find such essays in the back numbers of Biometrika[2] nor have the opportunity to search other references, *e.g.* Metron, Phil. Mag., Phil. Trans. &c., so it is tried to construct several new bimodal curves in the cases: (i) the distribution is in both sides unlimited, (ii) only in one side limited, (iii) in both sides limited.

§1. The differential equation[3] of frequency curves in the most general form shall be given by

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_0+a_1x+a_2x^2+\cdots\cdots}{b_0+b_1x+b_2x^2+\cdots\cdots},$$

which, however, is too extensive to be treated here. To get simply bimodal curves, it is sufficient to assume that they become minimum at origin, and maximum at two other (oppositely lying) points, so that the required D. E. reduces merely to

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2}, \tag{1}$$

where the numerator is to have real roots of different signs besides 0.

(i) *The case, where the distribution is in both sides unlimited.* In this case the denominator in (1) must not have any real root, so that, for the sake of brevity, we may assume the denominator simply to be 1:

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1) Although the present work is not so refined theoretically, the author aimed to utilize it as the stuff of exercise on mechanical computations for students: E. g. On the Decomposition of a bimodal Distribution into two normal Curves, T. Kudō and others, which, however, as has been not yet completed, would be published in the next number of this Journal.

2) Except the only one: Sui massimi delle curve dimorfiche, Dal Dr. Fernando de Helguero, Roma, Biometrika, vol. III (1904), p. 84,—which, however, does not go into details.

3) As well known, Pearson, starting from a problem of a game, adopts only the form $\frac{1}{y}\frac{dy}{dx}=\frac{a_0+x}{b_0+b_1x+b_2x^2}$, as the D. E. of his fundamental distributions.

$$\frac{1}{y}\frac{dy}{dx} = a_1 x + a_2 x^2 + a_3 x^3 \, .$$

We get, therefore,

$$\log \frac{y}{C} = \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \frac{1}{4} a_3 x^4 \, ,$$

i.e. $\quad y = C \exp \{ax^2 + bx^3 + cx^4\} \, .$

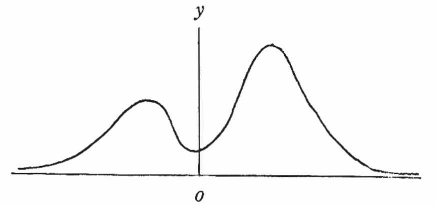

Fig. 1

Or, upon taking the mean value $\bar{x}$ as origin, and writing $x - \bar{x} = u$, we have

$$y = y_0 e^{\varphi(u)} \quad \text{with} \quad \varphi(u) = c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4 \, , \tag{2}$$

where, under assumption, $\varphi'(u)$ should have 3 real roots and $c_4 < 0$.

Now, in order to determine the constants in (2), we avail the moments formulas:

$$\int_{-\infty}^{\infty} y_0 e^{\varphi(u)} du = \mu_0 = 1 \, , \quad \int_{-\infty}^{\infty} y_0 u^n e^{\varphi(u)} du = \mu_n \quad (n = 1, 2, \cdots) \, , \tag{3}$$

in which numerical values of μ_n can be obtained from actual statistics, though the integrals themselves are not expressible in finite formes. So I make shift with the following treatment in a somewhat Pearson-like manner.

Firstly, integrating (3) by parts, we obtain

$$\mu_n = \left[y_0 u^{n+1} e^{\varphi(u)} / (n+1) \right]_{-\infty}^{\infty} - \frac{y_0}{n+1} \int_{-\infty}^{\infty} (c_1 u^{n+1} + 2c_2 u^{n+2} + 3c_3 u^{n+3} + 4c_4 u^{n+4}) e^{\varphi(u)} du \, ,$$

in which the integrated parts become zero as $c_4 < 0$ by assumption, and the remaining integral can be expressed in terms of moments, so as

$$(n+1)\mu_n = -c_1 \mu_{n+1} - 2c_2 \mu_{n+2} - 3c_3 \mu_{n+3} - 4c_4 \mu_{n+4} \, .$$

Putting here $n = 0, 1, 2, 3$ and observing that $\mu_0 = 1$, $\mu_1 = 0$, we get

$$\left.\begin{aligned} 0 \; + 2c_2\mu_2 + 3c_3\mu_3 + 4c_4\mu_4 &= -1 \, , \\ c_1\mu_2 + 2c_2\mu_3 + 3c_3\mu_4 + 4c_4\mu_5 &= \;\; 0 \;\; , \\ c_1\mu_3 + 2c_2\mu_4 + 3c_3\mu_5 + 4c_4\mu_6 &= -3\mu_2 \, , \\ c_1\mu_4 + 2c_2\mu_5 + 3c_3\mu_6 + 4c_4\mu_7 &= -4\mu_3 \, , \end{aligned}\right\} \tag{4}$$

from which the four unknowns c_1, c_2, c_3, c_4 can be determined. Substituted these values in (2) and (3), it gives

$$\int_{-\infty}^{\infty} \exp \{c_1 u + c_2 u^2 + c_3 u^3 + c_4 u^4\} du = \frac{1}{y_0} \, ,$$

whence by numerical computation the value of y_0 could be found. Since $c_4 < 0$ and the exponential tends rapidly to zero, we might execute the mechanical integration simply between $\pm L$ (numerically pretty large) instead of $\pm\infty$.

Example. A symmetrical distribution is given as in the second column of the following Table. Required to find the frequency curve.

x	y	x^2y	x^4y	x^6y
0	0.0690	0	0	0
±1	0.0819	0.082	0.08	0.1
±2	0.1210	0.484	1.94	7.7
±3	0.1553	1.398	12.58	113.2
±4	0.0952	1.523	24.37	389.9
±5	0.0120	0.300	7.50	187.5
±6	0.0001	0.004	0.13	4.7
sum	$1.0000=\mu_0$	$7.582=\mu_2$	$93.20=\mu_4$	$1406.2=\mu_6$

By reason of symmetry we may assume the distribution to be $y=y_0 \exp(c_2x^2+c_4x^4)$, and accordingly moments of odd order $=0$. Substituting the values of moments acquired from the above Table in (4), we find that $c_2=0.181$, $c_4=-0.010$ and directly $y_0=0.069$. Hence the required distribution is given by $y=0.069\exp(0.18x^2-0.01x^4)$, roughly.

As done above, the actual moments μ_n are usually computed by summations, but to speak more exactly, they need Sheppard's corrections, as well known, and this is so, not only for $n=2$ and 4, but also for $n>4$. In general, if the ficticious and true moment of order n about y-axis are ν_n' and ν_n respectively, *i.e.*

$$\nu_n'=\sum_i f_i x_i^n, \quad \nu_n=\int_{-\infty}^{\infty} f(x)x^n dx,$$

we have, in the case that $y=f(x)$ highly osculates x-axis,

$$\nu_n=\nu_n'-\frac{w^2}{3!4}n(n-1)\nu_{n-2}-\frac{w^4}{5!4^2}n(n-1)(n-2)(n-3)\nu_{n-4}-\cdots\cdots,$$

where $w=$ breadth of class taken in summation; and thus

$$\nu_0=\nu_0'=1,\quad \nu_1=\nu_1'=d,\quad \nu_2=\nu_2'-\frac{w^2}{12},\quad \nu_3=\nu_3'-\frac{w^2}{4}d,\quad \nu_4=\nu_4'-\frac{w^2}{2}\nu_2-\frac{w^4}{80},$$

$$\nu_5=\nu_5'-\frac{5}{6}w^2\nu_3-\frac{w^4}{16}d,\quad \nu_6=\nu_6'-\frac{5}{4}w^2\nu_4-\frac{3}{16}w^4\nu_2-\frac{w^6}{1792},$$

$$\nu_7=\nu_7'-\frac{7}{4}w^2\nu_5-\frac{7}{16}w^4\nu_3-\frac{w^6}{64}d,\quad \nu_8=\nu_8'-\frac{7}{3}w^2\nu_6-\frac{7}{8}w^4\nu_4-\frac{w^6}{16}\nu_2-\frac{w^8}{2304},$$

$$\nu_9=\nu_9'-3w^2\nu_7-\frac{63}{40}w^4\nu_5-\frac{3}{16}w^6\nu_3-\frac{w^8}{256}d,\ \cdots\cdots.$$

So the higher the order of moment, the larger the correction. In particular, if the origin be the mean, ν_n and ν_n' become μ_n and μ_n', the moments about mean, respectively. Notwithstanding the above correction formulas hold the same and become even simpler, because $\mu_1 = \mu_1' = d$ reduces to zero.

If we make Sheppard's correction in the preceding example, we obtain $\mu_2 = 7.499$, $\mu_4 = 89.47$, and $\mu_6 = 1293.0$ (though this is really of no use in the present case), so that the results become $c_2 = 0.210$, $c_4 = -0.012$ and $y = 0.069 \exp(0.21x^2 - 0.012x^4)$, thus pretty differ from those obtained before.

(ii) *The case, where the left handed side is limited, but the other side unlimited.* Assuming that the distribution extends from $x = -\gamma$ (negative) to $x = \infty$, the D. E. (1) can be written in the form

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_1 x + a_2 x^2 + a_3 x^3}{x+\gamma}\left(= \frac{\psi(x)}{x+\gamma}\right), \quad \gamma > 0, \; \psi(-\gamma) \neq 0, \; a_3 < 0, \; a_1 > 0.$$

This yields after integration

$$y = k \exp\{ax + bx^2 + cx^3\} \cdot (x+\gamma)^{-a\gamma},$$

where $a = a_1 - a_2\gamma + a_3\gamma^2 = \psi(-\gamma)/-\gamma \neq 0$,

$b = \frac{1}{2}(a_2 - a_3\gamma)$, $c = \frac{a_3}{3}$ and $-a\gamma > -1$.[4)]

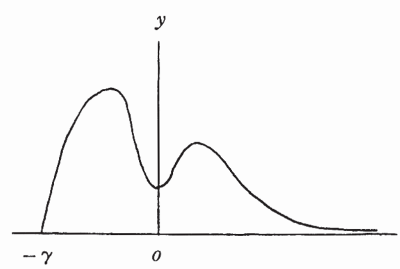

Fig. 2

Or, if we take $x = -\gamma$ as origin, and put $x = X - \gamma$, then the equation reduces to

$$\left.\begin{aligned} & y = KX^p e^{\varphi(X)}, \quad \varphi(X) = c_1 X + c_2 X^2 + c_3 X^3, \\ & \text{where} \\ & c_1 = a - 2b\gamma + 3c\gamma^2, \quad c_2 = b - 3c\gamma, \quad c_3 = c < 0 \quad \text{and} \quad p = -a\gamma > -1. \end{aligned}\right\} \quad (5)$$

Now taking the n-th moment about $X = 0$, we obtain

$$\nu_n = K\int_0^\infty X^{n+p} e^{\varphi(X)} dX$$
$$= \left[\frac{K}{n+p+1} X^{n+p+1} e^{\varphi(X)}\right]_0^\infty - \frac{K}{n+p+1}\int_0^\infty X^{n+p+1}(c_1 + 2c_2 X + 3c_3 X^2) e^{\varphi(X)} dX,$$

in which the integrated parts reduce to zero, and the remaining integral can be expressed in terms of moments of higher order, so as

$$(n+p+1)\nu_n + c_1\nu_{n+1} + 2c_2\nu_{n+2} + 3c_3\nu_{n+3} = 0.$$

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4) The assumption $-a\gamma > -1$ is made, so that the integration at $x = -\gamma$ may be possible. When $a < 0$, the curve really intersects x-axis at $x = -\gamma$; but if $a > 0$ (yet $a < 1/\gamma$), the negative root of $\frac{dy}{dx} = 0$ goes out from the interval $(-\gamma, \infty)$, and there y becomes imaginary, so that the curve degenerates J-shaped, having $x = -\gamma$ as asymptote. Suchlike gives rise, when p or $q < 0$ in (iii) below.

Putting here $n=0,1,2,3,4$, and in view of $\nu_0=1$, $\nu_1=d$ (mean), we get

$$\left.\begin{aligned}(p+1)\quad +c_1d+2c_2\nu_2+3c_3\nu_3&=0,\\(p+2)d+c_1\nu_2+2c_2\nu_3+3c_3\nu_4&=0,\\(p+3)\nu_2+c_1\nu_3+2c_2\nu_4+3c_3\nu_5&=0,\\(p+4)\nu_3+c_1\nu_4+2c_2\nu_5+3c_3\nu_6&=0,\\(p+5)\nu_4+c_1\nu_5+2c_2\nu_6+3c_3\nu_7&=0.\end{aligned}\right\}\qquad(6)$$

These ν_n's can be expressed in terms of moments μ_n's about the mean $X=d\,(=\nu_1)$, all of which are obtainable from the given statistics:

$$\nu_n=\mu_n+n\mu_{n-1}d+\frac{n(n-1)}{2}\mu_{n-2}d^2+\cdots\cdots+\frac{n(n-1)}{2}\mu_2d^{n-2}+d^n,$$

and thus

$$\left.\begin{aligned}&\nu_0=\mu_0=1,\quad \nu_1=d,\ (\mu_1=0),\quad \nu_2=\mu_2+d^2,\quad \nu_3=\mu_3+3\mu_2d+d^3,\\&\nu_4=\mu_4+4\mu_3d+6\mu_2d^2+d^4,\quad \nu_5=\mu_5+5\mu_4d+10\mu_3d^2+10\mu_2d^3+d^5,\\&\nu_6=\mu_6+6\mu_5d+15\mu_4d^2+20\mu_3d^3+15\mu_2d^4+d^6,\\&\nu_7=\mu_7+7\mu_6d+21\mu_5d^2+35\mu_4d^3+35\mu_3d^4+21\mu_2d^5+d^7.\end{aligned}\right\}\qquad(7)$$

These being substituted in (6), we obtain five equations which involve five unknowns p, c_1, c_2, c_3 and d. If d be regarded as known parameter for a while, so (6) can be looked as simultaneous linear equations of c_1, c_2, c_3 and p. Therefore, on solving any four, say the latter four of (6), and substituting their values in the first, we get an equation of higher degree about d. If its root $d=d_0$ be adequately chosen, all numerical values of c_1, c_2, c_3, p could be computed. Lastly the coefficient K would be obtained from

$$\int_0^\infty X^p \exp(c_1X+c_2X^2+c_3X^3)dX=\frac{1}{K}$$

by means of mechanical integration. The distribution function (5) is thus completely determined.

(iii) *The case, where both sides are limited.* Let the ends of the distribution be $-\gamma(<0)$ and $\delta(>0)$. The D. E. (1) may be written as

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_1x+a_2x^2+a_3x^3}{(x+\gamma)(\delta-x)}\left(=\frac{\psi(x)}{(x+\gamma)(\delta-x)}\right),$$

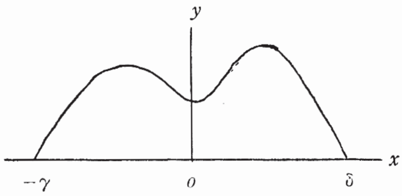

Fig. 3

where the quadratic $a_1+a_2x+a_3x^2$ should have two roots lying in $(-\gamma, o)$ and (o, δ) respectively, so that $a_1a_3<0$ and in fact

$a_1 > 0$ and $a_3 < 0$, as seen below[5]. Solving the D.E. we get

$$y = K(x+\gamma)^p(\delta-x)^q \exp\left[\{(\gamma-\delta)a_3-a_2\}x-\frac{1}{2}a_3x^2\right]=f(x),$$

where

$$p=\frac{\psi(-\gamma)}{l}, \quad q=-\frac{\psi(\delta)}{l}, \quad \psi(x)=a_1x+a_2x^2+a_3x^3, \quad l=\gamma+\delta \text{ (breadth).}$$

Taking the left end $x=-\gamma$ as origin, and writing $x+\gamma=X$, the equation becomes

$$y = cX^p(l-X)^q \exp(aX+bX^2). \tag{8}$$

Its moment about $X=0$ can be obtained as

$$\begin{aligned}\nu_n &= c\int_0^l X^{p+n}(l-X)^q \exp(aX+bX^2)dX \\ &= -\frac{c}{q+1}\Big[X^{p+n}(l-X)^{q+1}\exp(aX+bX^2)\Big]_0^l \\ &\quad +\frac{c}{q+1}\int_0^l \exp(aX+bX^2)X^{p+n}(l-X)^{q+1}\left\{\frac{p+n}{X}+a+2bX\right\}dX.\end{aligned}$$

Assumed that $p>-1$, $q>-1$[5], the integrated parts do vanish; and from the remaining integral, we have the following recurring formula

$$l(p+n)\nu_{n-1}=(n+1+p+q-al)\nu_n+(a-2bl)\nu_{n+1}+2b\nu_{n+2}.$$

On writing $\nu_0=1$, $\nu_1=d$, and

$$lp=A, \quad l=B, \quad p+q-al=C, \quad a-2bl=D, \quad 2b=E, \tag{9}$$

the above yields

$$(A+nB)\nu_{n-1}=(n+1+C)\nu_n+D\nu_{n+1}+E\nu_{n+2}.$$

Putting $n=1,2,\cdots,6$, we obtain the following six equations:

$$\begin{aligned}(A+B) &= (2+C)d+D\nu_2+E\nu_3, \\ (A+2B)d &= (3+C)\nu_2+D\nu_3+E\nu_4, \\ (A+3B)\nu_2 &= (4+C)\nu_3+D\nu_4+E\nu_5, \\ (A+4B)\nu_3 &= (5+C)\nu_4+D\nu_5+E\nu_6, \\ (A+5B)\nu_4 &= (6+C)\nu_5+D\nu_6+E\nu_7, \\ (A+6B)\nu_5 &= (7+C)\nu_6+D\nu_7+E\nu_8.\end{aligned} \tag{10}$$

5) If we investigate more closely the sign of $\psi(x)$ and the expansion of $y=f(x)$ at origin &c., we see that, when p, q are both positive, and moreover if $a_1>0$, (so $a_3<0$), then the curve becomes really bimodal in $(-\gamma, \delta)$, but if $a_1<0$ $(a_3>0)$, only unimodal in $(-\gamma, \delta)$, whereas, if p, q are both negative, the curve degenerates U-shaped, and bi- or uni- antimodal according as $a_1 \gtrless 0$. If p, q be one positive and one negative, then the curve falls into a distorted J-shape.

Whence, by the same reasoning as done in (ii), all values of A, B, C, D, E and d can be determined, and in succession from (9) all values of l, p, b, a, q, and finally the value of c by the numerical computation of

$$\int_0^l X^p(l-X)^q \exp\{aX+bX^2\}dX=\frac{1}{c}.$$

If the definite integrals of $y, yX, \cdots$ could be expressed in finite forms, the method would become far more facile.

§ **2.** Since I could not find Pearson's essay on the construction of bimodal distribution by means of superposition of unimodals, a conjectured plan of his method should be described below.

(i) In the case, that both sides are unlimited, anyone would suppose immediately the superposition of two normal distributions. Nevertheless, the actual analysis is a pretty troublesome[6].

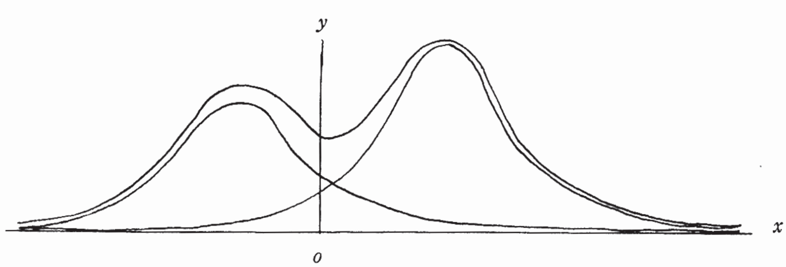

Fig. 4

The superposed frequency curve shall be

$$y=\frac{n_1}{\sqrt{2\pi}\,\sigma_1}\exp\left\{-\frac{(x-a_1)^2}{2\sigma_1^2}\right\}+\frac{n_2}{\sqrt{2\pi}\,\sigma_2}\exp\left\{-\frac{(x-a_2)^2}{2\sigma_2^2}\right\}, \qquad (1)$$

where $n_1+n_2=1$, if y is the probability density (or if y be the actual frequency, it shall be $n_1+n_2=N$, the actual total frequency).

To calculate the moment ν_n about origin, we write $t=(x-a_i)/\sigma_i$ $(i=1,2)$ and integrate yx^n between $\pm\infty$. Making use of formulas

$$\int_{-\infty}^{\infty} t^n e^{-t^2}dt=\begin{cases} 0, & \text{when } n=\text{odd}, \\ \Gamma\left(\frac{n+1}{2}\right), & \text{when } n=\text{even}, \end{cases}$$

we get easily the following results:

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[6] Kudô and others, *l.c.*

$$\nu_0 = 1 = n_1 + n_2, \quad \nu_1 = n_1 a_1 + n_2 a_2 \ (= d = \text{total mean}),$$
$$\nu_2 = n_1(a_1^2 + \sigma_1^2) + n_2(a_2^2 + \sigma_2^2), \quad \nu_3 = n_1 a_1(a_1^2 + 3\sigma_1^2) + n_2 a_2(a_2^2 + 3\sigma_2^2),$$
$$\nu_4 = n_1(a_1^4 + 6a_1^2\sigma_1^2 + 3\sigma_1^4) + n_2(a_2^4 + 6a_2^2\sigma_2^2 + 3\sigma_2^4),$$
$$\nu_5 = n_1 a_1(a_1^4 + 10a_1^2\sigma_1^2 + 15\sigma_1^4) + n_2 a_2(a_2^4 + 10a_2^2\sigma_2^2 + 15\sigma_2^4).$$

From these six equations we must determine six unknowns:

$$n_1, n_2, a_1, a_2, \sigma_1, \sigma_2.$$

Specially for symmetrical distribution, the origin being the total mean, we have $n_1 = n_2 = \frac{1}{2}$, $a_2 = -a_1$, $\sigma_1 = \sigma_2$, and consequently $a_1^2 + \sigma_1^2 = \mu_2$, $a_1^4 + 6a_1^2\sigma_1^2 + 3\sigma_1^4 = \mu_4$; whence we obtain

$$a_1^2 = \frac{1}{2}\sqrt{6\mu_2^2 - 2\mu_4}, \quad \sigma_1^2 = \mu_2 - \frac{1}{2}\sqrt{6\mu_2^2 - 2\mu_4}.$$

For the example treated in §1, we have $\mu_2 = 7.499$, $\mu_4 = 89.47$, so that $a_1^2 = 6.294$, $a_1 = \pm 2.51$, $\sigma_1^2 = 1.205$, $\sigma_1 = 1.10$ and as the required function

$$y = \frac{2.20}{\sqrt{2\pi}}\left[\exp\frac{-(x-2.51)^2}{2.41} + \exp\frac{-(x+2.51)^2}{2.41}\right],$$

although this representation is not so good compared with that in §1. To test the fitness more exactly, one ought to use χ^2– or ω^2–test.

(ii) If the distribution extends from $x = -\gamma$ to $x = \infty$, we may carry out the superposition of the curves from Pearson's type III

$$y = y_0\left(1 + \frac{x}{\gamma}\right)^{c\gamma} e^{-cx},$$

which have just alike ends. Or, on translating the origin into the left end, and writing $x + \gamma = X$,

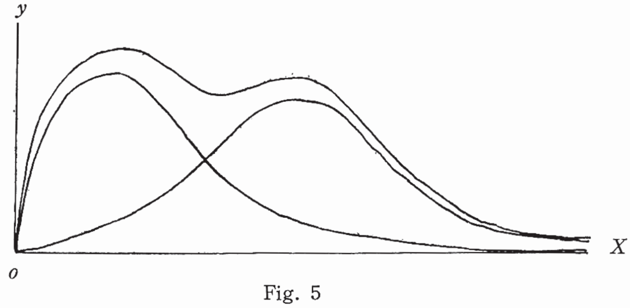

Fig. 5

$$y = kX^p e^{-cX} \quad (c > 0, \ p = c\gamma > -1).$$

Hence the required bimodal curve shall be

$$y = kX^p e^{-cX} + k'X^{p'} e^{-c'X} \quad (c, c' > 0, \ p, p' > -1). \tag{2}$$

The n–th moment of the first component about $X = 0$ is

$$k\int_0^\infty X^{n+p} e^{-cX} dX = k\int_0^\infty \left(\frac{t}{c}\right)^{n+p} e^{-t}\frac{dt}{c} \quad (cX = t)$$
$$= \frac{k}{c^{n+p+1}}\Gamma(n+p+1).$$

Hence, if we put

$$\nu_0 = 1 = \frac{k\Gamma(p+1)}{c^{p+1}} + \frac{k'\Gamma(p'+1)}{c'^{p'+1}} = K + K',$$

the n-th moment of (2) becomes

$$\nu_n = \frac{(p+1)(p+2)\cdots\cdots(p+n)}{c^n} K + \frac{(p'+1)\cdots\cdots(p'+n)}{c'^n} K'.$$

By putting here $n = 0, 1, 2, \cdots, 6$, we obtain seven equations, which contain p, p', c, c' and K, K'. As done in (1.7), ν_n's can be expressed linearly in μ_n's, so that, on solving thus obtained seven equations, we can evaluate the seven unknowns, *i.e.* besides $\nu_1 = d$ the above six unknowns, and lastly k, k' from

$$k = \frac{Kc^{p+1}}{\Gamma(p+1)}, \quad k' = \frac{K'c'^{p'+1}}{\Gamma(p'+1)}$$

by use of the Table of gamma function.

Otherwise, if the given distribution make strong contact with X-axis at both ends, we may replace the foregoing by Pearson's type V, and thus consider

$$y = kX^{-q} \exp\{-\gamma/X\} + k'X^{-q'} \exp\{-\gamma'/X\}, \tag{3}$$

where $\gamma, \gamma' > 0$ and q, q' are assumed to be sufficiently large, so that the moments of pretty higher order still may exist. Consequently, so far as $n < q-1$ is,

$$\nu_n = k\gamma^{n+1-q}\Gamma(q-n-1) + k'\gamma'^{n+1-q'}\Gamma(q'-n-1).$$

And if we put

$$\nu_6 = k\gamma^{7-q}\Gamma(q-7) + k'\gamma'^{7-q'}\Gamma(q'-7) = L + L',$$

the others can be written as

$$\begin{aligned}
\nu_5 &= L\gamma^{-1}(q-7) + L'\gamma'^{-1}(q'-7), \\
\nu_4 &= L\gamma^{-2}(q-7)(q-6) + L'\gamma'^{-2}(q'-7)(q'-6), \\
\nu_3 &= L\gamma^{-3}(q-7)(q-6)(q-5) + \cdots\cdots, \\
\nu_2 &= L\gamma^{-4}(q-7)(q-6)(q-5)(q-4) + \cdots\cdots, \\
\nu_1 &= L\gamma^{-5}(q-7)(q-6)(q-5)(q-4)(q-3) + \cdots\cdots, \\
\nu_0 &= L\gamma^{-6}(q-7)\cdots\cdots(q-2) + L'\gamma'^{-6}(q'-7)\cdots\cdots(q'-2).
\end{aligned}$$

Again, upon expressing ν_n's in μ_u's by (1.7), we can compute from these seven equations the seven unknowns $L, L', \gamma, \gamma', q, q'$ and $\nu_1 = d$, and lastly k, k' from

$$k = Lk^{q-7}/\Gamma(q-7), \quad k' = L\gamma'^{q'-7}/\Gamma(q'-7)$$

by means of the Table of gamma function.

(iii) In case that both ends are limited, we may refer to Pearson's type I, and assume

$$\left.\begin{array}{l} y = kX^p(l-X)^q + k'X^{p'}(l-X)^{q'}, \\ \quad (\text{all } p, q, p', q' > -1,\ l > 0). \end{array}\right\} \qquad (4)$$

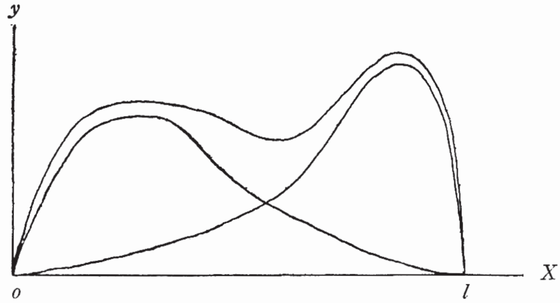

Fig. 6

The area under the curve is

$$\nu_0 = 1 = k\int_0^l X^p(l-X)^q dX + k'\int_0^l X^{p'}(l-X)^{q'} dX$$
$$= kl^{p+q+1}B(p+1,\ q+1) + k'l^{p'+q'+1}B(p'+1,\ q'+1),$$

while the n-th moment

$$\nu_n = k\int_0^l X^{n+p}(l-X)^q dX + k'\int_0^l X^{n+p'}(l-X)^{q'} dX$$
$$= kl^{n+p+q+1}B(n+p+1,\ q+1) + k'l^{n+p'+q'+1}B(n+p'+1,\ q'+1).$$

Hence, if A and A' be two components in ν_0, we obtain

$$\nu_n = \frac{(p+1)(p+2)\cdots\cdots(p+n)}{(p+q+2)\cdots\cdots(p+q+n+1)} l^n A + \frac{(p'+1)\ \cdots\cdots(p'+n)}{(p'+q'+2)\cdots\cdots(p'+q'+n+1)} l^n A'.$$

Here letting $n = 0, 1, \cdots, 6$, we obtain, as before, seven equations containing seven unknowns, A, A', p, p', q, q' and $d(=\nu_1)$; whence all unknowns can be evaluated, and lastly k and k' from

$$k = A/l^{p+q+1}B(p+1,\ q+1), \quad k' = A'/l^{p'+q'+1}B(p'+1,\ q'+1)$$

by means of the Table of Beta function.