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NOTES ON FINITE SEMIGROUPS AND DETERMINATION[0] OF SEMIGROUPS OF ORDER 4

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In the present paper we shall give some remarks about finite semigroups and shall determine all types of semigroups of order 4. The computation is performed by use of the elementary method [1][1], the results of semigroups of order 2, 3 [2], and our developed theories of semigroups. A general method of determination of finite semigroups (of an arbitrary order) is not yet found out.

§ 1. Unipotent semigroups.

1 Unipotent semigroups. In the previous paper [3] we argued some properties of finite unipotent semigroups. Furthermore we argued them from more general standpoint in another article [4] by Clifford and Miller's theory [5].

We mean by a zero-semigroup a unipotent semigroup whose idempotent is a two-sided zero 0. Apart from zero-semigroups of order n, unipotent semigroups of order n are determined in such a way as following, if zero-semigroups of order $m<n$ are all given.

A group G of order g, $g<n$, a zero-semigroup Z of order m where $m=n-g+1$, and a homomorphism f of $M=G\cup Z'$ onto G determine uniquely a unipotent semigroup of order n, greatest group of which is G [4].

Z' symbols the set of all elements of Z except a zero, and M is the union of G and Z'.

All unipotent semigroups of order 4 other than zero-semigroups are $u-11\sim u-19$[2], in which I. (2)[3] is the class of types $g=2$, I. (3) types $g=3$, and I. (4) groups.

2 Zero-semigroups. Let a be an element of a semigroup S. If there exists $x\in S$ such that $ax=a$, a is called an left-invariant (or l-invariant)

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[0] This research was sponsored, in part, by MIKI-KORAKUKAI. See Addendum at the end.

[1] The number in the bracket [ ] shows the number of References appearing at the end.

[2] It is an individual number of a type in the table at the end.

[3] It is a number of a class of types in the same table.

element. Right-invariant (r–invariant) element is likewise defined. Here we denote by Z a finite zero-semigroup.

Lemma 1. *Z contains no l–invariant (r–invariant) element except* 0.

Proof. Suppose that there is an l–invariant element a different from 0. Set $X=\{x;\ x\in Z,\ ax=a\}$. It is seen that X is a subsemigroup of Z and does not contain 0; whence no idempotent lies in X because Z is a zero-semigroup. This conflicts with the fact that a finite semigroup contains at least one idempotent.

We introduce two orderings into Z: left ordering and right ordering. $a \underset{l}{\gtrsim} b$ means that either $a=b$ or $a=bx$ for some $x\in Z$, and $a \underset{r}{\gtrsim} b$ means that either $a=b$ or $a=yb$ for some $y\in Z$.

Lemma 2. *The two orderings are all partial orderings.*

Proof. Reflexivity and transitivity are clear. We shall prove anti-symmetry. If $a=bx$ and $b=ay$ for some x and y, then $a=a(yx)$. But, from Lemma 1, it follows that $a=0$ and so $a=b=0$. Similar as to right ordering.

Due to each of the two ordering, Z is a partly ordered set having 0 as greatest element. Now an element a is called an l–minimal element if $a \underset{l}{\gtrsim} b$ for no $b\neq a$. Likewise an r–minimal element is defined. Since Z is finite, minimal elements exist. Specially when a is least, a is called l–least (r–least) element.

Lemma 3. *If a is an l–minimal element, then a is also r–minimal, and vice versa.*

Proof. If a is not l–minimal, then $a=bx$ for some $b, x\in Z$; so $a \underset{r}{\gtrsim} x$, that is, a is not r–minimal. The proof of the converse is similar. We notice that a may be supposed to be distinct from 0, because the case of a trivial zero-semigroup $Z=\{0\}$ is out of consideration.

Lemma 4 *Let Z be a finite zero-semigroup. The following conditions are all equivalent.*

(i) *Z has an l–least element.*

(ii) *Z has an r–least element.*

(iii) *Z forms a chain with respect to the l–ordering.*

(iv) *Z forms a chain with respect to the r–ordering.*

(v) *Z is a power semigroup.*

Proof.

$$(\text{ii}) \rightleftarrows (\text{i}) \rightarrow (\text{v}) \begin{cases} \nearrow (\text{iii}) \rightarrow (\text{i}) \\ \searrow (\text{iv}) \rightarrow (\text{ii}) . \end{cases}$$

(i) $\rightleftarrows$ (ii): obvious by Lemma 3. (i) $\rightarrow$ (v). Let a be the l–least element. Every $x \neq a$ is written $x = ay$, that is, the v–order is $n-1$ where n is the d–order of Z. According to [6], Z is a power semigroup. (v) $\rightarrow$ (iii) and (v) $\rightarrow$ (iv) are easily proved. (iii) $\rightarrow$ (i) and (iv) $\rightarrow$ (ii) are clear.

We denote by $[a]$ the power subsemigroup generated by a:

$$[a] = \{a^i ; i = 1, 2, \dots\} .$$

Now we define the third ordering in Z: $a \geqq b$ means that $[a] \subset [b]$. This is called the power-ordering (p–ordering).

Lemma 5. *$a \geqq b$ if and only if $a = b^n$ for some positive integer n.* (see [7])

Lemma 6. *The ordering $\geqq$ is a partial ordering.*

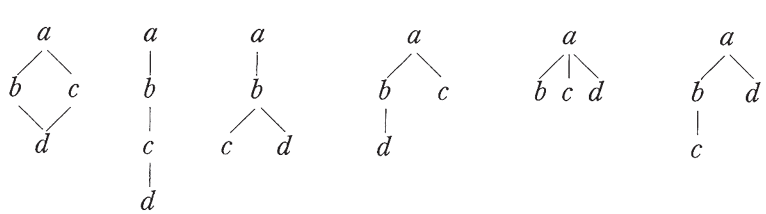
Proof. We shall show anti-symmetry. If $a \geqq b$ and $b \geqq a$ i. e., $b = a^n$ and $a = b^m$, then $a = a^{mn}$ which leads to $a = 0$ by Lemma 1, and hence $a = b = 0$.

Lemma 7. *$a \geqq b$ implies $a \underset{l}{\gtrsim} b$ and $a \underset{r}{\gtrsim} b$.*

Proof. By Lemma 5, $a = b^n = bb^{n-1} = b^{n-1}b$.

We shall construct, as an example, all types of zero-semigroups Z of order 4 by the aid of the above lemmas.

All types of partly ordered set of order 4, which has a greatest element, are shown as following.



0–1 0–2 0–3 0–4 0–5 0–4′

These become naturally semilattices.

Under consideration of Lemmas 3, 4 and 7, the following table designates all possible triple combinations chosen among them as l–, r– and p–orderings in Z and all types of Z deduced from the combinations. By Lemma 4, 0–1 cannot be taken as l–ordering (r–ordering).

The class [I].

l-ordering	0–5	0–2		0–4	0–4	0–3	0–3			0–4	0–4′
r-ordering	0–5	0–2		0–4	0–3	0–4	0–3			0–4′	0–4
p-ordering	0–5	0–1	0–2	0–4	0–4	0–4	0–3	0–4′	0–5	0–5	0–5
type of Z	u–1	u–2	none	u–3	u–4	u–4′	u–5 u–7 u–7′ u–8	u–6	u–9	u–10	u–10′

where u–4′, u–10′ are dually isomorphic with u–4, u–10 respectively.

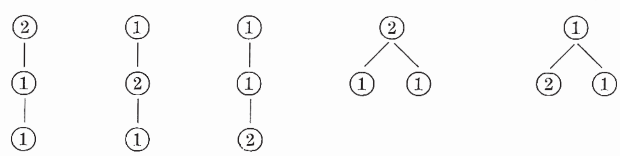
§ 2. Commutative semigroups.

According to [8], a finite commutative semigroup S is decomposed into the class sum of mutually disjoint unipotent subsemigroups and the quotient set forms a semilattce. Let L be a semilattice obtained in greatest decomposition of S by which $S=\bigcup_{i=1}^{n} S_i$.

By the types of L and S_i, all types of a non-unipotent commutative semigroup S of order 4 is classified into the following. Below, ⓘ symbols a unipotent commutative semigroup of order i ($i=1, 2, 3$)

(1) Semilattice,

(2) 2–1–1 type.



l–1 l–2 l–3 l–4 l–5 ,

(3) 2–2 type

l–6 ,

(4) 3–1 type

l–7 l–8 .

On the other hand, the types of ② and ③ are

② $\begin{vmatrix} A & B \\ B & A \end{vmatrix}$ U$_2$–1 $\begin{vmatrix} A & A \\ A & A \end{vmatrix}$ U$_2$–2 ,

③ $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix}$ U$_3$ 1 $\begin{vmatrix} A & B & B \\ B & A & A \\ B & A & A \end{vmatrix}$ U$_3$–2 $\begin{vmatrix} A & B & A \\ B & A & B \\ A & B & A \end{vmatrix}$ U$_3$–3 $\begin{vmatrix} A & A & A \\ A & A & A \\ A & A & B \end{vmatrix}$ U$_3$–4 $\begin{vmatrix} A & A & A \\ A & A & A \\ A & A & A \end{vmatrix}$ U$_3$–5 .

Computating all types of S by elementary operations in each ease, we have

(1) Semilattice: VI. (1), i. e., c–1 $\sim$ c–5,

(2) 2–1–1 type: VI. (2),

② \ L	l–1	l–2	l–3	l–4	l–5
U_2–1	c–6	c–10	c–14	c–15	c–18
U_2–2	c–7 c–8 c–9	c–11 c–12	c–13	c–16 c–17	c–19

(3) 2–2 type: VI. (3),

lower \ upper	U_2–1	U_2–2
U_2–1	c–20 c–21	c–23 c–24
U_2–2	c–22	c–25 c–26

(4) 3–1 type: VI. (4),

L \ ③	U_3–1	U_3–2	U_3–3	U_3–4	U_3–5
l–7	c–27	c–28 c–29	c–30 c–31	c–32 c–33	c–34 c–35 c–36 c–37
l–8	c–38	c–39	c–40	c–42	c–41

§ 3 Non-commutative semigroups.

Generally a semigroup S is able to be decomposable to a commutative semigroup, i. e.,

$$S=\bigcup_{\alpha=1}^{k} S_\alpha, \qquad S_\alpha \frown S_\beta=\phi \qquad (\alpha \neq \beta),$$

and for α and β there is γ such that $S_\alpha S_\beta \subset S_\gamma$ and $S_\beta S_\alpha \subset S_\gamma$. It is proved that there exists a greatest decomposition of S to a commutative semigroup [9]. The meaning of greatest decomposition is due to [8].

Let C be a commutative semigroup obtained in greatest commutativity decomposition of S. We classify types of S into various classes according as the type of C.

1. *Commutativity-indecomposable semigroup.*

When C is composed of only one element, S is called commutativity-indecomposable (c–indecomposable).

Lemm 8. *If S is c–indecomposable, then $SS=S$.*

Proof. Suppose that $SS \subsetneqq S$.
Let $A = SS$ and $B = S - SS$, then $S = A \cup B$; this is a commutaitivity-decomposition.

Lemma 9. *If S is c–indecomposable semigroup of order* 4, *then S has no tow-sided ideal of order* 2, 3.

Proof. (i) The proof of having no ideal of order 3.
Suppose that S has an ideal A of order 3 and let $S = A \cup \{x\}$. Since $AS \subset A$ and $SA \subset A$, we have $x^2 = x$ by Lemma 8; whence the decomposition of S, $S = A \cup \{x\}$, gives commutativity.

(ii) The proof of having no ideal of order 2.

Let $S = \{a, b, c, d\}$. Suppose that S has an ideal $\{a, b\}$. Since $SS = S$ and S has no ideal of order 3, the four cases are considered·

F–1

	a	b	c	d
a				
d				
c			d	c
d				

F–2

	a	b	c	d
a				
b				
c				c
d			d	

F–3

	a	b	c	d
a				
b				
c				d
d				c

F–4

	a	b	c	d
a				
b				
c			d	
d				c

But, by an elementary theory,
we have

d	c
c	d

from F–1,

c	c
d	d

from F–2,

c	d
d	c

from F–3,

d	d
d	c

from F–4.

This shows that S is c–decomposable such that $S = \{a, b\} \cup \{c, d\}$, contradicting with the assumption. Hence S has no ideal of order 2.
Consequently we get

Theorem. *A c–indecomposable semigroup S of order* 4 *is completely simple* [10].

Proof. By Lemma 9, S has no ideal of order $\geqq 2$. Since a finite semigroup is simple, S is proved to be completely simple.

The theorem makes it possible to establish types of commutativity-indecomposable semigroup as II, i–1, i–2.

2. *c–decomposable semigroups.*
The types of C are as follows.

	A	B
A	A	A
B	A	B

D_2–1

	A	B
A	A	B
B	B	A

D_2–2

	A	B
A	A	A
B	A	A

D_2–3

A	B	C
B	C	A
C	A	B

D_3–1

A	B	B
B	A	A
B	A	A

D_3–2

A	B	A
B	A	B
A	B	A

D_3–3

A	A	A
A	A	A
A	A	B

D_3–4

A	A	A
A	A	A
A	A	A

D_3–5

A	A	A
A	B	C
A	C	B

D_3–6

A	A	A
A	B	C
A	C	A

D_3–7

A	B	A
B	A	B
A	B	C

D_3–8

A	A	A
A	B	B
A	B	C

D_3–9

A	A	A
A	B	A
A	A	C

D_3–10

A	A	A
A	A	A
A	A	C

D_3–11

A	A	A
A	B	B
A	B	B

D_3–12

After complicated computations we have the following result.

2–2 type: III,

C	D_2–1	D_2–2	D_2–3
S	(1) 2·2–1∼2·2–7	(2) 2·2–8	*none*

3–1 type: IV,

A	*a a a* *a a a* *a a a*	*a b c* *a b c* *a b c*	*a b a* *a b a* *a b a*
S	(1) 3·1–1 3·1–2	(2) 3·1–3∼3·1–8	(3) 3·1–9∼3·1–13

There is no type of S in other cases than above 3 types of A.

2–1–1 type: V,

C	D_3–1	D_3–2	D_3–3	D_3–4	D_3–5	D_3–6	D_3–7	D_3–8	D_3–9	D_3–10	D_3–11	D_3–12
S	*none*	*none*	*none*	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

We notice that dually isomorphic non-commutative semigroups are omitted in the table at the end. Thus we have obtained 194 types of semigroups of order 4.

Finally I express my heartfelt thanks to Mr. M. Yamamura, Mr. T. Akazawa and Mr. R. Shibata for their devotional works of the complicated computation.

Semigroups of order 4.

I. Unipotent

(1)

1

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	a

u-1

2

a	a	a	a
a	a	a	a
a	a	a	b
a	a	b	c

u-2

3

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	b

u-3

4

a	a	a	a
a	a	a	a
a	a	a	a
a	a	b	b

u-4

5

a	a	a	a
a	a	a	a
a	a	b	b
a	a	b	b

u-5

6

a	a	a	a
a	a	a	a
a	a	b	b
a	a	b	a

u-6

7

a	a	a	a
a	a	a	a
a	a	b	b
a	a	a	b

u-7

8

a	a	a	a
a	a	a	a
a	a	b	a
a	a	a	b

u-8

9

a	a	a	a
a	a	a	a
a	a	a	b
a	a	b	a

u-9

10

a	a	a	a
a	a	a	a
a	a	a	a
a	a	b	a

u-10

(2)

11

a	b	b	b
b	a	a	a
b	a	a	a
b	a	a	a

u-11

12

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	c

u-12

13

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	a

u-13

14

a	b	a	b
b	a	b	a
a	b	a	b
b	a	b	a

u-14

15

a	b	a	b
b	a	b	a
a	b	a	b
b	a	b	c

u-15

(3)

16

a	b	c	a
b	c	a	b
c	a	b	c
a	b	c	a

u-16

17

a	b	c	b
b	c	a	c
c	a	b	a
b	c	a	c

u-17

(4)

18

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

u-18

19

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

u-19

II. Commutativity-indecomposable

20

a	b	c	d
a	b	c	d
a	b	c	d
a	b	c	d

i-1

21

a	b	a	b
a	b	a	b
c	d	c	d
c	d	c	d

i-2

III. Decomposable, 2–2 type

(1)

22

a	b	a	a
a	b	a	a
a	b	c	d
a	b	c	d

2·2–1

23

a	b	a	a
a	b	b	b
a	b	c	d
a	b	c	d

2·2–2

24

a	b	a	b
a	b	a	b
a	b	c	d
a	b	c	d

2·2–3

25

a	a	a	a
a	a	a	a
a	b	c	d
a	b	c	d

2·2–4

26

a	a	a	a
a	a	b	b
a	a	c	d
a	a	c	d

2·2–5

27

a	b	a	a
a	b	a	a
a	b	c	c
a	b	d	d

2·2–6

28

a	b	a	a
a	b	b	b
a	b	c	c
a	b	d	d

2·2–7

(2)

29

a	b	c	d
a	b	c	d
c	d	a	b
c	d	a	b

2·2–8

IV. Decomposable, 3–1 type

(1)

30

a	a	a	a
a	a	a	b
a	a	a	c
a	a	a	d

3·1–1

31

a	a	a	a
a	a	a	b
a	a	a	a
a	a	c	d

3·1–2

(2)

32

a	b	c	a
a	b	c	a
a	b	c	a
a	b	c	d

3·1–3

33

a	b	c	a
a	b	c	b
a	b	c	a
a	b	c	d

3·14

34

a	b	c	a
a	b	c	b
a	b	c	c
a	b	c	d

3·1–5

35

a	b	c	d
a	b	c	d
a	b	c	d
d	d	d	d

3·1–6

36

a	b	c	a
a	b	c	a
a	b	c	a
a	b	c	a

3·1–7

37

a	b	c	a
a	b	c	a
a	b	c	b
a	b	c	a

3·1–3

(3)

38

a	b	a	a
a	b	a	a
a	b	a	c
a	b	a	d

3·1–9

39

a	b	a	a
a	b	a	b
a	b	a	c
a	b	a	d

3·1–10

40

a	b	a	a
a	b	a	b
a	b	a	a
a	b	c	d

3·1–11

41

a	b	a	a
a	b	a	a
a	b	a	a
a	b	c	d

3·1–12

42

a	b	a	b
a	b	a	b
a	b	a	b
a	b	c	d

3·1–13

V. Decomposable, 2–1–1 type

(1)

43

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	c

2·1·1–1

(2)

44

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	a

2

45

a	b	a	b
a	b	a	b
a	b	a	b
a	b	a	b

3

(3)

46

a	b	a	a
a	b	b	b
a	b	c	d
a	b	d	c

4

47

a	b	a	b
a	b	b	a
a	b	c	d
a	b	d	c

5

48

a	b	b	b
a	b	b	b
a	b	c	d
a	b	d	c

6

49

a	a	a	a
a	a	a	a
a	b	c	d
a	b	d	c

7

(4)

50

a	b	a	a
a	b	b	a
a	b	c	d
a	b	d	a

8

51

a	b	b	b
a	b	b	b
a	b	c	d
a	b	d	b

9

52

a	a	a	a
a	b	c	d
a	b	c	d
a	d	d	a

10

53

a	a	a	a
a	a	b	a
a	a	c	d
a	a	d	a

11

54

a	a	a	a
a	b	c	c
a	c	a	a
a	d	a	a

12

(5)

55

a	a	c	a
a	a	c	a
c	c	a	c
a	b	c	d

13

56

a	b	a	a
b	a	b	b
a	b	c	d
a	b	c	d

14

57

a	b	b	a
b	a	a	b
b	a	a	b
a	b	c	d

15

(6)

58

a	b	a	a
a	b	b	b
a	b	c	c
a	b	c	d

16

59

a	b	b	a
a	b	b	b
a	b	c	c
a	b	c	d

17

60

a	b	b	b
a	b	b	b
a	b	c	c
a	b	c	d

18

61

a	a	a	a
a	a	b	b
a	a	c	c
a	a	c	d

19

62

a	a	a	a
a	a	b	b
a	a	c	c
a	b	c	d

20

63

a	a	a	a
a	a	a	a
a	a	c	c
a	b	c	d

21

64

a	a	a	a
a	b	c	b
a	b	c	c
a	b	c	d

22

65

a	a	a	a
a	b	c	c
a	b	c	c
a	b	c	d

23

66

a	a	a	a
a	b	b	b
a	b	b	b
a	b	c	d

24

67

a	a	a	a
a	b	b	b
a	b	c	d
a	b	c	d

25

(7)

68

a	b	a	a
a	b	b	a
a	b	c	a
a	b	a	d

26

69

a	b	b	a
a	b	b	a
a	b	c	a
a	b	b	d

27

70

a	b	b	b
a	b	b	b
a	b	c	b
a	b	b	d

28

71

a	a	a	a
a	a	b	a
a	a	c	a
a	a	a	d

29

72

a	a	a	a
a	a	b	a
a	a	c	a
a	b	a	d

30

73

a	a	a	a
a	a	b	a
a	a	c	a
a	b	b	d

31

74

a	a	a	a
a	b	b	a
a	c	c	a
a	a	a	d

32

(8)

75

a	b	a	a
a	b	a	a
a	b	a	a
a	b	a	d

33

76

a	b	a	b
a	b	a	b
a	b	a	b
a	b	a	d

34

77

a	b	a	a
a	b	a	b
a	b	a	a
a	b	a	d

35

78

a	a	a	a
a	a	a	a
a	a	a	a
a	b	a	d

36

79

a	a	a	a
a	a	a	a
a	a	a	a
a	b	b	d

37

80

a	a	a	a
a	a	a	a
a	a	c	d
a	a	c	d

38

(9)

81

a	b	b	b
a	b	b	b
a	b	c	c
a	b	c	c

39

82

a	b	a	a
a	b	b	b
a	b	c	c
a	b	c	c

40

83

a	a	a	a
a	a	b	b
a	a	c	c
a	a	c	c

41

84

a	a	a	a
a	b	c	b
a	b	c	b
a	b	c	b

42

VI. Commutative, non-unipotent

(1) Semilattice

85

a	a	a	a
a	b	b	b
a	b	c	c
a	b	c	d

c–1

86

a	a	a	a
a	b	b	b
a	b	c	b
a	b	b	d

c–2

87

a	a	a	a
a	b	a	b
a	a	c	a
a	b	a	d

c–3

88

a	a	a	a
a	b	a	b
a	a	c	c
a	b	c	d

c–4

89

a	a	a	a
a	b	a	a
a	a	c	a
a	a	a	d

c–5

(2) Semilattice-decomposable 2–1–1 type

90

a	b	a	a
b	a	b	b
a	b	c	c
a	b	c	d

c–6

91

a	a	a	a
a	a	b	b
a	b	c	c
a	b	c	d

c–7

92

a	a	a	a
a	a	a	b
a	a	c	c
a	b	c	d

c–8

93

a	a	a	a
a	a	a	a
a	a	c	c
a	a	c	d

c–9

94

a	a	a	a
a	b	c	b
a	c	b	c
a	b	c	d

c–10

95

a	a	a	a
a	b	b	b
a	b	b	c
a	b	c	d

c–11

96

a	a	a	a
a	b	b	b
a	b	b	b
a	b	b	d

c–12

97

a	a	a	a
a	b	b	b
a	b	c	c
a	b	c	c

c–13

98

a	a	a	a
a	b	b	b
a	b	c	d
a	b	d	c

c–14

99

a	b	a	a
b	a	b	b
a	b	c	a
a	b	a	d

c–15

100

a	a	a	a
a	a	b	a
a	b	c	a
a	a	a	d

c–16

101

a	a	a	a
a	a	a	a
a	a	c	a
a	a	a	d

c–17

102

a	a	a	a
a	b	c	a
a	c	b	a
a	a	a	d

c–18

103

a	a	a	a
a	b	b	a
a	b	b	a
a	a	a	d

c–19

(3) Semilattice-decomposable 2–2 type

104

a	b	a	a
b	a	b	b
a	b	c	d
a	b	d	c

c–20

105

a	b	a	b
b	a	b	a
a	b	c	d
b	a	d	c

c–21

106

a	b	a	a
b	a	b	b
a	b	c	c
a	b	c	c

c–22

107

a	a	a	a
a	a	b	b
a	b	c	d
a	b	d	c

c–23

108

a	a	a	a
a	a	a	a
a	a	c	d
a	a	d	c

c–24

109

a	a	a	a
a	a	b	b
a	b	c	c
a	b	c	c

c–25

110

a	a	a	a
a	a	a	a
a	a	c	c
a	a	c	c

c–26

(4) Semilattice-decomposable 3–1 type

111

a	b	c	a
b	c	a	b
c	a	b	c
a	b	c	d

c–27

112

a	b	b	a
b	a	a	b
b	a	a	b
a	b	b	d

c–28

113

a	b	b	a
b	a	a	b
b	a	a	c
a	b	c	d

c–29

114

a	b	a	a
b	a	b	b
a	b	a	c
a	b	c	d

c–30

115

a	b	a	a
b	a	b	b
a	b	a	a
a	b	a	d

c–31

116

a	a	a	a
a	a	a	b
a	a	b	c
a	b	c	d

c–32

117

a	a	a	a
a	a	a	a
a	a	b	a
a	a	a	d

c–33

118

a	a	a	a
a	a	a	a
a	a	a	a
a	a	a	d

c–34

119

a	a	a	a
a	a	a	a
a	a	a	c
a	a	c	d

c–35

120

a	a	a	a
a	a	a	b
a	a	a	b
a	b	b	d

c–36

121

a	a	a	a
a	a	a	b
a	a	a	c
a	b	c	d

c–37

122

a	a	a	a
a	b	c	d
a	c	d	b
a	d	b	c

c–38

123	124	125	126
a a a a	a a a a	a a a a	a a a a
a b c c	a b c b	a b b b	a b b b
a c b b	a c b c	a b b b	a b b b
a c b b	a b c b	a b b b	a b b c
c-39	*c*-40	*c*-41	*c*-42

Addendum

Although all types of semigroups of order 4 were already found out by Mr. M. Yamamura & the writer in 1953, we have computed them once more by utilizing the new theories. We have heard from Prof. E. Hewitt, University of Washington, that Prof. G. E. Forsythe, Unviversity of California, is computing them by a very large electronic computer.

August, 1954.

References

[1] By the elementary method we mean the method of determination of semigroups by using Theorem 1 & 3 in [2].

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[4] T. Tamura, Note on unipotent inversible semigroups, Kodai Math. Semi. Rep, No. 3, October, 1954. pp. 93–95.

[5] A. H. Clifford & D. D. Miller, Semigroups having zeroid elements, Amer. Jour. of Math., Vol. LXX, No. 1, 1948, pp. 117–125.

[6] See Theorem 9 at p. 19 in [3].

[7] See Lemma 2 in the paper: T. Tamura, On a monoid whose submonoids form a chain, in this Journal.

[8] T. Tamura, On decompositions of a commutative semigroup, Kodai Math. Semi. Rep. (unpublished).

[9] The proof will be dealed with in another paper.

[10] Rees, D., On semigroups, Proc. Cambridge Phil. Soc. Vol. 36, 1940, pp. 387–400.