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# ON A MONOID WHOSE SUBMONOIDS FORM A CHAIN[0)]

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## § 1. Introduction.

Generally the set $\mathfrak{S}$ of submonoids of a monoid[1)] $M$ constitutes a complete lattice.[2)] Although it is of course that the structure of $\mathfrak{S}$ is given by that of $M$, some property of $M$ is characterized by a property of $\mathfrak{S}$. This paper is concerned with the determination of all types of a monoid whose all submonoids form a chain. We shall call such a monoid a $\Gamma$-monoid. In case when $M$ is a finite group, the problem is solved by R. Baer [1] *i.e.*,

**Lemma 1.** *The lattice formed by all submonoids of a finite group $G$ is a chain if and only if $G$ is a cyclic group of prime power order.*

In the present paper, it will be concluded that if a $\Gamma$-monoid $M$ is finite, $M$ is a certain power monoid of order $n$, where $p^m \leq n \leq p^m+2$, and $p$ is a prime number, and if $M$ is infinite, $M$ is a limit group of finite cyclic groups of prime power order.

## § 2. Preliminaries.

In the below Lemmas 2 and 3 we assume $M$ to be a monoid. Let us denote by $[a]$ a submonoid of $M$ generated by only an element $a \in M$, *i.e.*,

$$[a] = \{a^i;\ i = 1, 2, 3, \cdots\}.$$

If $[a]$ is infinite (finite), then the element $a$ is said to be an element of infinite (finite) order or an infinite (finite) element. We define a quasi-ordering $a \leq b$ as $[a] \subset [b]$.

**Lemma 2.** *$a \leq b$ if and only if $a = b^n$ for some positive integer $n$.*

**Proof.** If $a = b^n$ for some $n$, then $a^m = (b^n)^m = b^{nm} \in [b]$ for every $m$. Therefore $[a] \subset [b]$. The converse is clear by the definition.

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1) The "monoid" and "submonoid" are synonyms of the "semigroup" and "subsemigroup" respectively. cf. N. Bourbaki; Structure algebriques.

2) We shall consider even the empty set as a submonoid.

Let $\bar{M}$ be a quotient set got by introducing into $M$ the equivalence relation $a \sim b$ defined as $a \leqq b$ and $b \leqq a$. $\bar{M}$ is a partly ordered set.

**Lemma 3.** *There is an element b different from a such that $a \sim b$ if and only if $[a]$ is a finite cyclic group of order $n \geqq 3$.*

**Proof.** If $a \sim b$ as well as $a \neq b$, then $a = b^k$ and $b = a^m (k \neq 1,\ m \neq 1)$ by Lemma 2, and we have $a = a^{km}$ where $km \geqq 4$. It follows that $a$ is of finite order and it belongs to the greatest group $G$ of $[a]$ (see [2]). Hence we get $[a] = G$. Next, supposing that $a = a^t$, $t = 2$ or 3, it is readily led that $a = a^2 = b$. Therefore the order of $G$ is at least 3. Conversely if $[a]$ is a cyclic group of order $n \geqq 3$, there is a positive integer $m$ such that $1 < m < n$ and $m$ is relatively prime to $n$. Then $a^m \neq a$ and $a \sim a^m$. Thus the proof of the lemma has been completed.

Hereafter we assume $\mathfrak{S}$ to be a chain, in other words, $M$ to be a $\Gamma$-monoid, and $\mathfrak{S}$ is represented as

$$\mathfrak{S} = \{S_\gamma;\ \gamma \in \Lambda\}$$

where the set $\Lambda$ of suffixes is a chain, and has 0 as the least element and $\xi$ as the greatest, *i.e.*, $S_0 = \phi$, $S_\xi = M$, and $S_\gamma \subset S_\zeta$ for $\gamma < \zeta$.

**Lemma 4.** *Every submonoid of a $\Gamma$-monoid is a $\Gamma$-monoid.*

**Proof.** Let $S$ be a submonoid of $M$ and $\mathfrak{T}$ be the set of all submonoids of $S$. Of course $\mathfrak{T} \subset \mathfrak{S}$. The ordering in $\mathfrak{S}$ is preserved in $\mathfrak{T}$.

**Lemma 5.** *The homomorphic image $M' = f(M)$ of a $\Gamma$-monoid $M$ by the homomorphism $f$ is a $\Gamma$-monoid.*

**Proof.** Let $S_\gamma'$ and $S_\zeta'$ be submonoids of $M'$, and let $S_\gamma$ and $S_\zeta$ their inverse images by $f$ respectively. By the assumption, either $S_\gamma \subset S_\zeta$ or $S_\zeta \subset S_\gamma$; and so evidently $f(S_\gamma) \subset f(S_\zeta)$ or $f(S_\zeta) \subset f(S_\gamma)$. Thus $M'$ is proved to be a $\Gamma$-monoid.

**Lemma 6.** *If $M$ is a $\Gamma$-monoid, every element of $M$ is of finite order. Namely $[a]$ is a finite power monoid.*

**Proof.** Suppose that there is an infinite element $a \in M$. By Lemma 4, $[a]$ is a $\Gamma$-monoid. But we see that $[a]$ has two incomparable submonoids

$$[a^2] = \{a^{2i};\ i = 1, 2, 3, \cdots\}, \quad [a^3] = \{a^{3i};\ i = 1, 2, 3, \cdots\};$$

this is contradictory with the assumption. Hence every element is of finite order.

## § 3. Type of chain.

We denote by $\mathfrak{S}'$ the set of all power submonoids of $M$. $\mathfrak{S}'$ is a subchain of $\mathfrak{S}$ admitting a chain $\Gamma''$ as an index set, and

$$\mathfrak{S}' = \{S_\gamma;\ \gamma \in \Gamma''\},\quad \Gamma'' \subset \Gamma$$

where $S_\gamma \subset S_\zeta$ for $\gamma < \zeta$.
We easily have

**Lemma 7.** *Every $S_\gamma$ is finite,*

**Lemma 8.** *$M'$ is order-isomorphic with $\Gamma''$.*

The following lemma is remarkable.

**Lemma 9.** *The ordinal number of $\Gamma''$ is not greater than the first infinite ordinal number $\omega$.*

**Proof.** When $\Gamma''$ is finite, it is evident that $\Gamma''$ has finite ordinal number. We shall discuss as to the case that $\Gamma''$ is infinite. Let $\sigma$ be any element of any subset $\Sigma'$ of $\Gamma''$. By Lemmas 7 and 8, the number of the elements of $\Sigma'$ which lie before $\sigma$ is finite; and so $\Sigma'$ has a least element. In other words, $\Gamma''$ is a well-ordered set, the ordinal number of which we denote by $\gamma$. Since $\Gamma''$ is infinite, $\omega \leqq \gamma$. Next, suppose that $\omega+1 \leqq \gamma$, then it follows that $S_\omega$ is infinite. This contradicts with Lemma 7. Henceforth we have $\gamma = \omega$.

According to the above lemmas, all elements of $\mathfrak{S}'$ may be generally denoted as follows:

if $M$ is finite, $\phi = S_{-1} \subsetneqq S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_\eta$,
if $M$ is infinite, $\phi = S_{-1} \subsetneqq S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_\gamma \subsetneqq S_{\gamma+1} \subsetneqq \cdots$

where $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$ for no $S_\delta \in \mathfrak{S}'$ $(\gamma = -1, 0, 1, 2, \cdots)$.

An increasing sequence $\{S_\gamma\}$ of power submonoids of $M$ where there is no power submonoid $S_\delta$ such that $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$ is called a full chain of power submonoids of $M$.

**Lemma 10.** *An increasing sequence $\{S_\gamma\}$ is a full chain of power submonoids if and only if any element of $S_{\gamma+1} - S_\gamma$ generates $S_{\gamma+1}$.*

**Proof.** Suppose $\{S_\gamma\}$ is a full chain of $M$. Set $S_\gamma = [a]$ and $T = S_{\gamma+1} - S_\gamma$. Obviously $[x] \subset S_{\gamma+1}$ for any $x \in T$, and we get $[a] \subsetneqq [x] \subset S_{\gamma+1}$. Hence $[x] = S_{\gamma+1}$. Conversely if any element of $S_{\gamma+1} - S_\gamma$ generates $S_{\gamma+1}$, it is seen that there is no $S_\delta$ such that $S_\gamma \subsetneqq S_\delta \subsetneqq S_{\gamma+1}$.

From Lemma 10 we obtain easily the following

**Lemma 11.** *If $M$ is a $\Gamma$-monoid, then there exists a full chain*

$$[a_0]\subset[a_1]\subset\cdots\subset[a_\gamma]\subset\cdots$$

*of at most countable power submonoids such that* $M=\bigcup_{\gamma=0}^{\infty}[a_\gamma]$.

A full chain $\{[a_\gamma]\}$ satisfying $M=\bigcup_{\gamma=0}^{\infty}[a_\gamma]$ is called a basic chain of $M$. The below lemma is worthy of notice.

**Lemma 12.** *If A monoid $M$ has a basic chain $\{[a_\gamma]\}$, any proper submonoid of $M$ is a power monoid.*

**Proof.** Let $S$ be any proper submonoid of $M$. There exists greatest $\bar{\gamma}$ of $\gamma$ such that $[a_\gamma]\subset S$. For, if not so, $[a_\gamma]\subset S$ for every $\gamma$, and so $M=S$. Now, since $[c]\subset S$ for every $c\in S$, we have $c\in[c]\subset[a_{\bar{\gamma}}]$; and $S\subset[a_{\bar{\gamma}}]$. Combining it with $[a_{\bar{\gamma}}]\subset S$, we get $S=[a_{\bar{\gamma}}]$.

Thus it is concluded that every submonoid of a $\Gamma$-monoid $M$ is no other than a power monoid which forms a full chain of $M$.

The following theorems are immediately obtained.

**Theorem 1.** *If $M$ is a $\Gamma$-monoid, the ordinal number of $\mathfrak{S}$ is not greater than $\omega+1$, and every proper submonoid of $M$ is a finite power monoid.*

**Theorem 2.** *A monoid $M$ is a $\Gamma$–monoid if and only if $M$ has a basic chain.*

As special case we have

**Lemma 13.** *If $M$ is a $\Gamma$-monoid as well as a group, then $[a]$ is a prime power cyclic group for every $a\in M$. Moreover the order of $[a]$ is a power of the same prime number.*

**Proof.** Let $a$ be any element different from the unit $e$ of $M$. Of course $[a]$ is finite. We let $n$ be the order of $a$:

$$a^n=e \qquad (n>1).$$

For every $m\geq n$, $a^m$ belongs to the cyclic group, the greatest group $G_0$ of $[a]$ (see [2]).

From
$$a^n=a^{2n}=e,$$
$$aa^{n-1}=a^{n+1}a^{n-1}.$$

Since $M$ is a group, we get $a=a^{n+1}$ by multiplying the both sides by the inverse of $a^{n-1}$. Hence $a\in G_0$, that is to say, $[a]$ is a cyclic group. It is owing to Lemma 1 that $[a]$ is a prime power group. The latter half of the lemma is readily shown.

## § 4. Type of difference monoid.

**Lemma 14.** *A $\Gamma$-monoid is unipotent inversible* [3].

**Proof.** If there exist distinct idempotents $a$ and $b$ in $M$, then $\{a\}$ and $\{b\}$ are incomparable submonoids of $M$. This conflicts with the assumption. Therefore $M$ is unipotent. By Lemma 6, any element $a$ is represented as $a^n = aa^{n-1} = e$ for some $n > 1$; that is, $M$ is inversible.

According to [2] [3], $G = Me$ is the greatest group of $M$. We denote by $M^*$ the difference monoid [4] of $M$ modulo $G$. Then $M^*$ is a $\Gamma$-zero-monoid [2] and every element of $M^*$ is of finite order by Lemmas 5 and 6.

**Lemma 15.** *Let $Z$ be a $\Gamma$-zero-monoid. Every element of $Z$ is of order*[3] *at most* 3.

**Proof.** If there is an element $x$ of order 4 in $Z$,

$$[x] = \{x, x^2, x^3, 0\}, \quad x^4 = 0,$$

contains two incomparable submonoids

$$A = \{0, x^2\} \quad \text{and} \quad B = \{0, x^3\},$$

contradicting with the definition of a $\Gamma$-monoid. If there is an element $x \in Z$ is of order $n > 4$, then a power zero-monoid $[x]$ is homomorphic onto a power zero-monoid $C = \{X, X^2, X^3, X^4 = 0\}$ [2] and the submonoids $S_1$ and $S_2$ which correspond to $\{0, X^2\}$ and $\{0, X^3\}$ respectively are incomparable.

**Theorem 3.** *A zero-monoid $Z$ is a $\Gamma$-monoid if and only if $Z$ is a power zero-monoid of order*[4] *at most* 3.

**Proof.** Suppose that $Z$ is a $\Gamma$-zero-monoid. If the number of elements of a zero-monoid $Z$ is no less than 4 or infinite, Lemma 15 makes it possible for us to find different elements $x$ and $y$ having equal order $m$ where $m$ is 2 or 3. Then it is seen that $[x]$ and $[y]$ are incomparable submonoids of $Z$. Hence $Z$ is composed of at most 3 elements. Conversely we shall prove that a zero-monoid of order at most 3 is a $\Gamma$-monoid.

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3) By the order $n$ of an element $x$ of a zero-monoid, we mean such $n$ that $x^n=0$ and $x^m \neq 0$ for $1 \leq m < n$.

4) We mean the order of a monoid $M$ the number of elements of $M$.

Since a zero-monoid of order 2 is nothing but

| | $0$ | $a$ |
|---|---|---|
| $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ |

,

the proof of this case is trivial. Using the theory of a finite zero-monoid [2] [5], it is proved that zero-monoids of order 3 have two types as the following:

| | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $0$ |

,

| | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $a$ |

.

The former is neither a power monoid nor a $\Gamma$-monoid for $\mathfrak{S}$ is

$$\begin{array}{ccc} & \{0, a, b\} & \\ \{0, a\} & & \{0, b\} \\ & \{0\} & \end{array}.$$

The latter is not only a power-monoid but a $\Gamma$-monoid. In fact, $\mathfrak{S}$ is

$$\begin{array}{c} \{0, a, b\} \\ | \\ \{0, a\} \\ | \\ \{0\}. \end{array}$$

Thus we have completed the proof.

By Theorem 3, the difference monoid $M^*$ of $M$ modulo $G$ has been verified to consist of at most 3 elements.

## § 5. Infinite $\Gamma$-monoid.

Now we shall determine the type of the infinite $\Gamma$-monoid in this paragraph.

**Lemma 16.** *An infinite $\Gamma$-monoid is a group.*

**Proof.** Let $M$ be an infinite $\Gamma$-monoid, and $G$ be this greatest group. Suppose that $G \subsetneqq M$, then $G$ is finite by Theorem 1, and the difference monoid of $M$ modulo $G$ is finite by Theorem 3. Accordingly $M$ is finite; this contradicts with the assumption. This shows that $G = M$.

As a result of Theorem 2, Lemmas 2 and 3, the structure of an infinite $\Gamma$-monoid is clarified in the following manner.

At first, we shall explain a "limit group of groups". There is given an increasing sequence of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_\gamma \subset \cdots$$

and isomorphisms $\phi_\delta^\gamma$ of $G_\gamma$ into $G_\delta$ ($\gamma < \delta$) satisfying $\phi_\zeta^\delta \phi_\delta^\gamma = \phi_\zeta^\gamma$. Let $G$ be the union of $G_\gamma$ ($\gamma = 0, 1, 2, \cdots$): $G = \bigcup_\gamma G_\gamma$ and let $\bar{G}$ be the quotient set of $G$ obtained by identifying

$$x \in G_\gamma \quad \text{with} \quad y = \phi_\delta^\gamma(x) \in G_\delta.$$

The product $xy$ of $x$ and $y$ in $\bar{G}$ is defined as the product of $x$ and $y$ in a certain group $G_\gamma$ containing them. Then $\bar{G}$ is clearly a group. $\bar{G}$ is called a limit group of $\{G_\gamma;\ \phi_\delta^\gamma\}$.

Now, in an infinite $\Gamma$-monoid $M$ there is a basic chain $\{[a_\gamma]\}$ such that $[a_\gamma]$ is a cyclic group of prime power order $p^\gamma$ and

$$M = \bigcup_{\gamma=0}^{\infty} S_\gamma$$

where $$S_\gamma = [a_\gamma], \quad a_0 = e, \quad a_\gamma = a_{\gamma+1}^p \qquad (\gamma = 0, 1, 2, \cdots).$$

It is readily seen that $M$ is a limit group of $\{S_\gamma;\ \phi_\delta^\gamma\}$ where $\phi_\delta^\gamma$ is a mapping of each element of $S_\gamma$ into itself in $S_\delta$.

Conversely, if we are given cyclic groups $S_\gamma$ of order $p^\gamma$ ($\gamma = 0, 1, 2, \cdots$), an isomorphism $\phi_\delta^\gamma$ of $S_\gamma$ into $S_\delta$ is uniquely determined and it holds $\phi_\zeta^\delta \phi_\delta^\gamma = \phi_\zeta^\gamma$. Accordingly we can consider the limit group of $\{S_\gamma;\ \phi_\delta^\gamma\}$. Then then sequence

$$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_\gamma \subset \cdots$$

is a full chain of power submonoids of $M$, because there is no power submonoid $S_\delta$ between $S_\gamma$ and $S_{\gamma+1}$ ($\gamma = 0, 1, \cdots$). Consequently, by Theorem 2, $M$ is a $\Gamma$-monoid and

$$M = \bigcup_{\gamma=0}^{\infty} [a_\gamma]$$

where $a_\gamma$ is a generator of $S_\gamma$, or $\{[a_\gamma]\}$ is a basic chain of $M$.

**Theorem 4.** *An infinite $\Gamma$-monoid is a limit group of cyclic groups $S_\gamma$ ($\gamma = 0, 1, \cdots$) of order $p^\gamma$ where $p$ is a prime number, and vice versa.*

**Corollary.** *An infinite $\Gamma$-monoid is isomorphic with the additive group $E$ of modulo 1 as follows.*

$$E = \left\{ \frac{m}{p^n} ;\ m = 0, 1, 2, 3, \cdots, p^n - 1;\ n = 0, 1, 2, 3, \cdots \right\}.$$

## § 6. Finite $\Gamma$-monoids.

Finally we shall establish all types of finite $\Gamma$–monoids.

**Lemma 17.** *A finite $\Gamma$–monoid is a power monoid.*

**Proof.** The full chain of power monoids of a finite $\Gamma$–monoid $M$ ceases at finite terms:

$$[a_0] \subset [a_1] \subset \cdots \subset [a_n] \quad \text{and} \quad M = \bigcup_{i=0}^{n} [a_i].$$

Take any $x \in M$, then $x \in [a_t] \subset [a_n]$ for some $t \leqq n$. Hence $M \subset [a_n]$; we have $M = [a_n]$.

Since the greatest group $G$ of a finite $\Gamma$–monoid $M$ is a cyclic group of prime power order $p^m$, the types of $M$ is limited to the three, because of Theorem 3,

(1) $M$ is a power monoid of order $p^m$ *i.e.*, $M$ is a cyclic group,

(2) $M$ is a power monoid of order $p^m+1$,

(3) $M$ is a power monoid of order $p^m+2$,

where $p^m$ is the order of $G$.

Hereafter we shall investigate the types of (2) and (3).

**Lemma 18.** *Let $p$ be a prime number. A power monoid $M$ of order $p^m+1$, whose greatest group $G$ is of order $p^m$, is a $\Gamma$–monoid.*

**Proof.** Let $a$ be a generator of $M$. It is not hard to see

$$M = \{a, a^2, a^3, \cdots, a^{p^m}, a^{p^m+1}\},$$

where
$$a^2 = a^{p^m+2} \quad \text{and} \quad G = \{a^2, a^3, \cdots, a^{p^m+1}\}.$$

Since a submonoid containing $a$ coincides with $M$, we see easily that $M$ is a $\Gamma$–monoid.

As to (3), we divide the cases into the two: $p \neq 2$ and $p = 2$.

**Lemma 19.** *Let $p$ be a prime number $\neq 2$. A power monoid $M$ of order $p^m+2$, whose greatest group $G$ is of order $p^m$, is a $\Gamma$–monoid.*

**Proof.** We may prove that a submonoid $S$ containing $a^2$ is nothing but $\{a^2\} \cup G$. Given any $\mu_0 \geqq 3$,

$$2\nu \equiv \mu_0 \qquad (mod.\ p^m)$$

has a solution $\nu$. This shows that all elements of $G$ are generated by $a^2$. Hence $S = \{a^2\} \cup G$.

**Lemma 20.** *A power monoid $M$ of order $2^m+2$, whose greatest group is of order $2^m$, is not a $\Gamma$–monoid.*

**Proof.** In order that the congruence equation

$$2\nu \equiv \mu_0 \qquad (mod.\ 2^m)$$

has a solution, $\mu_0$ must be even. Let $S$ be a power submonoid generated by $a^2$, then $S \frown G = \{a^4, a^6, \cdots, a^{2^m+2}\}$. It follows that $S$ and $G$ are incomparable.

Putting together Lemmas 18–20, we have

**Theorem 5.** *A finite monoid $M$ is a $\Gamma$-monoid if and only if $M$ is a power monoid of order $n$ satisfying two conditions*:

(1) *the greatest group $G$ of $M$ is of prime power order $p^m$*,

(2) $p^m \leqq n \leqq p^m+1$ *if* $p=2$,
$p^m \leqq n \leqq p^m+2$ *if* $p \neq 2$.

Thus we have established all types of finite or infinite $\Gamma$-monoids.

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