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NOTES ON GENERAL ANALYSIS (IV)

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In preceding papers, we discussed the variation of extended $M(r)$[1] of analytic functions in complex Banach spaces. In this note, we take up first the order of entire function in complex Banach spaces and discuss it by using extended $M(r)$ in §1.

In 1937, Professor A. E. Taylor[2] pointed out that the theorems of Weierstrass and Picard were invalid generally and showed the existence of poles of infinite orders in complex Banach spaces. Here, we investigate the isolated singular point of analytic functions in §2.

Finally, in §3, the extended lemma of Schwarz[3] will be applied to various cases which will show us the convenience of treating analytic functions in abstract spaces.

§1. The order of entire functions

Let $E_0, E_1, \cdots, E_n$ be complex Banach spaces. An E_2–valued function $f(x)$ defined in a domain (which is open and connected) in E_1 is called analytic if it is strongly continuous and admits G–differential. An E_2–valued function $h_n(x)$ defined in E_1 is called a homogeneous polynomial of degree n, if it is analytic and satisfies $h_n(\alpha x) = \alpha^n h_n(x)$ for an arbitrary complex number α.

Definition 1. *An E_2–valued function $f(x)$ defined in E_1 is called an entire function if it is analytic on whole spaces.*

Definition 2. *Put* $\rho_1 = \overline{\lim_{r\to\infty}} \dfrac{\log\log M(r)}{\log r}$, *where* $M(r) = \operatorname{Sup}_{\|x\|=r} \|f(x)\|$ *and* $f(x)$ *is an entire function.*

Definiton 3. *Put* $\rho_2 = \operatorname{Sup}_{\|x\|=1} \overline{\lim_{r\to\infty}} \dfrac{\log\log M(r,x)}{\log r}$[4], *where* $M(r,x) = \operatorname{Sup}_{|\alpha|=r} \|f(\alpha x)\|$ *for an arbitrary point x on the set $\|x\| = 1$ and an entire function $f(x)$.*

Theorem 1. *If a radius of bound of entire function $f(x)$ is finite, then $\rho_1 = +\infty$. If a radius of bound of en entire function $f(x)$ is infinite, then*

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\sup_{||x||=1} ||h_n(x)||}},$$

where $f(x) = \sum_{n=0}^{\infty} h_n(x)$ *and* $h_n(x)$ *is a homogeneous polynomial of degree* n.

Proof. If a radius of bound λ of en entire function $f(x)$ is finite, $M(r) = +\infty$[5] for $r > \lambda$. Then, $\dfrac{\log\log M(r)}{\log r} = +\infty$, for sufficiently larger r such that $r > \lambda$ and $r > 1$. This shows that

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = +\infty .$$

If $\lambda = +\infty$, then $\overline{\lim_{r\to\infty}} \sqrt[n]{\sup_{||x||=1} ||h_n(x)||} = 0$. Let ε be an arbitrary positive number, then there exists a positive number r_0 such that

$$M(r) < e^{r^{\rho_1+\varepsilon}},$$

for $r \geqslant r_0$, from the definition of ρ_1.
On the other hand, for an arbitrary point x on the set $||x|| = 1$, we have $f(x) = \sum_{n=0}^{\infty} h_n(x)$, since $f(x)$ is analytic on whole space E_1. Then

$$\sup_{||x||=1} ||h_n(x)|| \leqslant \sup_{||x||=1} \frac{M(r, x)}{r^n} \leqslant \frac{M(r)}{r^n} \leqslant \frac{e^{r^{\rho_1+\varepsilon}}}{r^n}. \qquad \cdots\cdots\cdots\cdots (1)$$

Since $\dfrac{e^{r^{\rho_1+\varepsilon}}}{r^n}$ takes its minimum at r_1 which satisfies $r_1^{\rho_1+\varepsilon} = \dfrac{n}{\rho_1+\varepsilon}$, the inequality (1) holds for such r_1 if n is sufficiently large. Thus we have

$$\sup_{||x||=1} ||h_n(x)|| \leqslant \frac{e^{r_1^{\rho_1+\varepsilon}}}{r_1^n} = \left(\frac{e(\rho_1+\varepsilon)}{n}\right)^{\frac{n}{\rho_1+\varepsilon}}.$$

Taking the logarithum of the parts of two sides of the inequality,

$$\rho_1 + \varepsilon \geqslant \frac{\log n - \log e(\rho_1+\varepsilon)}{\dfrac{1}{n} \log \dfrac{1}{\sup_{||x||=1} ||h_n(x)||}}.$$

Since $\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{||x||=1} ||h_n(x)||} = 0$, $\rho_1 + \varepsilon \geqslant \overline{\lim_{n\to\infty}} \dfrac{\log n}{\dfrac{1}{n} \log \dfrac{1}{\sup_{||x||=1} ||h_n(x)||}}$,

and then we have

$$\rho_1 \geqslant \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\sup_{||x||=1} ||h_n(x)||}}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

because ε is an arbitrary positive number.

Put $$\rho = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|}}, \quad \text{then} \quad \operatorname{Sup}_{\|x\|=1} \|h_n(x)\| < \left(\frac{1}{n}\right)^{\frac{n}{\rho+\varepsilon}}, \qquad \cdots\cdots (3)$$

for an arbitrary positive number ε and $n \geqslant n_0(\varepsilon)$.

Then we have $\overline{\lim_{n\to\infty}} \sqrt[n]{\operatorname{Sup}_{\|x\|=1} \|h_n(x)\|} = 0$, and we see that $\sum_{n=0}^{\infty} h_n(x)$ is an entire function. Since $\left(\frac{1}{n}\right)^{\frac{1}{\rho+\varepsilon}} r < \frac{1}{2}$ for $n \geqslant n(r)$,

$$\operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \, r^n < \frac{1}{2^n} \quad \text{for} \quad n \geqslant \max. (n(r), n_0(\varepsilon)).$$

Then,

$$M(r) \leqslant \sum_{n=0}^{\infty} \operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \, r^n = \sum_{n=0}^{n(r)-1} \operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \, r^n + \sum_{n(r)}^{\infty} \operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \, r^n.$$

Put $c(r) = \operatorname{Sup}_{n\geqslant 0} (\operatorname{Sup}_{\|x\|=1} \|h_n(x)\| \, r^n)$, then $M(r) \leqslant n(r) \, c(r) + \frac{1}{2^{n(r)-1}}$.

Since $c(r) \leqslant e^{\frac{r^{\rho+\varepsilon}}{(\rho+\varepsilon)e}}$ (which is the maximum of $\left(\frac{1}{n}\right)^{\frac{n}{\rho+\varepsilon}} r^n$ for $n \geqslant 0$ and a sufficiently large r) from (3) and $n(r) \leqslant (2r)^{\rho+\varepsilon}$ from $\left(\frac{1}{n}\right)^{\frac{1}{\rho+\varepsilon}} r < \frac{1}{2}$,

$$M(r) \leqq (2r)^{\rho+\varepsilon} e^{\frac{r^{\rho+\varepsilon}}{(\rho+\varepsilon)e}} + \frac{1}{2}.$$

Then we have $$\overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} \leqslant \rho + \varepsilon,$$

for an arbitrary positive number ε. From (2) and (4), $\rho = \overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r}$. This completes the proof.

Theorem 2. $$\rho_2 = \operatorname{Sup}_{\|x\|=1} \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\|h_n(x)\|}}.$$

Proof. Since $f(x)$ is an entire function, we have

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r, x)}{\log r} = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\|h_n(x)\|}},$$

as well as $M(r)$, for an arbitrary point x on the set $\|x\| = 1$. Then we have

$$\rho_2 = \operatorname{Sup}_{\|x\|=1} \overline{\lim_{r\to\infty}} \frac{\log \log M(r, x)}{\log r} = \operatorname{Sup}_{\|x\|=1} \overline{\lim_{n\to\infty}} \frac{n \log n}{\log \dfrac{1}{\|h_n(x)\|}}.$$

Theorem 3. $\rho_2 \leqslant \rho_1$.

Proof. Since $M(r, x) \leqslant M(r)$, $\frac{\log \log M(r, x)}{\log r} \leqslant \frac{\log \log M(r)}{\log r}$.

and we have $\rho_2 \leqslant \rho_1$.

§ 2. Singular points of analytic functions.

A point x is called a singular point of $f(x)$, when $f(x)$ is not analytic in any neighbourhood of x. A singular point x is called an isolated singular point, if $f(x)$ is analytic in a neighbourhood of x droping itself. In this chapter, we research the state of an isolated singular point.

Definition 4. *Let an E_2-valued function $R_n(x)$ be analytic in $0<\|x\|<\infty$ in E_1. If $R_n(\alpha x)=\frac{1}{\alpha^n}R_n(x)$ for any complex number α, then $R_n(x)$ is called a homogeneous rational function of degree n.*

Theorem 4. *If $f(x)$ is analytic in $0<\|x\|<R$, then*

$$f(x)=\sum_{n=0}^{\infty}h_n(x)+\sum_{n=1}^{\infty}R_n(x)\,,$$

where $h_n(x)$ is a homogeneous polynomial of degree n and $R_n(x)$ is a homogeneous rational function of degree n.

Proof. Let x be an arbitrary point in $0<\|x\|<R$ and α be a complex number. Then $f(\alpha x)$ is an analytic function of α in $0<|\alpha|<\frac{R}{\|x\|}$ and we have

$$f(x)=\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha-1}d\alpha-\frac{1}{2\pi i}\int_{C'}\frac{f(\alpha x)}{\alpha-1}d\alpha\,,$$

where C is a circle such that $|\alpha|=r(>1)$ and C' is a circle $|\alpha|=r'(<1)$. Since the series $\sum_{n=0}^{\infty}\frac{f(x\alpha)}{\alpha^{n+1}}$ and $\sum_{n=1}^{\infty}f(\alpha x)\alpha^n$ converge uniformly respectively on C and C', we have

$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha\right)+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C'}f(\alpha x)\alpha^n d\alpha\right)\\&=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha\right)+\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C}f(\alpha x)\alpha^n d\alpha\right),\end{aligned}$$

because $f(\alpha x)\alpha^n$ is analytic as to α for $0<|\alpha|<\frac{R}{\|x\|}$.
Put $P_n(x)=\frac{1}{2\pi i}\int_C\frac{f(\alpha x)}{\alpha^{n+1}}d\alpha$ and $R_n(x)=\frac{1}{2\pi i}\int_C f(\alpha x)\alpha^{n-1}d\alpha$, then $P_n(x)$ is as usual a homogeneous polynomial of degree n by the uniformity of the integral and the theorem of Zorn[6]. From the analyticity of $f(\alpha x)$ in $0<\|x\|<R$ and the uniformity of convergence of the integral $\frac{1}{2\pi i}\int_C f(\alpha x)\alpha^n d\alpha$ we know that $R_n(x)$ is analytic in $0<\|x\|<\infty$, appealing to also the analytic continuation. Let ξ be an arbitrary complex number and x be an arbitrary point in E_1. Then we can take as C a circle with radius r which satisfies $0<r|\xi|\cdot\|x\|<R$. Then

$$R(\xi x) = \frac{1}{2\pi i}\int_{|\alpha|=r} f(\alpha\xi x)\alpha^{n-1}d\alpha\,.$$

Put $\xi\alpha = \beta$, then

$$R(\xi x) = \frac{1}{2\pi i}\int_{C'} f(\beta x)\frac{\beta^{n-1}}{\xi^{n-1}}\cdot\frac{1}{\xi}\,d\beta$$

$$= \frac{1}{\xi^n}\cdot\frac{1}{2\pi i}\int_{C} f(\beta x)\beta^{n-1}d\beta$$

$$= \frac{1}{\xi^n}R(x)\,,$$

where C' is a circle with radius r' which satisfies $r' = r|\xi|$. Thus we see that $R_n(x)$ is a homogeneous rational function of degree n.

Now, we must research that which space has an isolated singular point. A set of points $\{x_1+\alpha y_1,\ x_2+\beta y_2)\}$, where points (x_1, x_2), (y_1, y_2) are fixed and α, β are arbitrary complex numbers, is called a 2–dimensional plane. If the intersection of a set Γ and an arbitrary 2–dimensional plane is connected or null set, Γ is called 2–dimensionally connected.

Lemma[7]. *Let $f(x_1, x_2)$ in $E_1\times E_2$ to E_3 be analytic on the boundary Γ of a bounded domain Δ of $E_1\times E_2$, where Γ is 2–dimensionally connected. Then $f(x_1, x_2)$ is analytic in Δ.*

Theorem 5. *If $f(x)$ has an isolated singular point, then E_0 is the one dimensional space with respect to complex numbers.*

Proof. By the axiom of Zermelo, complex Banach space is considered as a well ordered set. Then we can find a set S of elements which are linearly independent by the transfinite induction. If S does not consist of only an element, S is divided an element and others which span a subspace E_1 and E_2 separately. Then we have $E_0 = E_1 + E_2$ as a direct sum of E_1 and E_2. If we assume that 0 is an isolated singular point of $f(x)$ to simplify the notation, $f(x)$ is analytic on $||x|| = \rho$, for sufficiently small positive number ρ. Appealing to Lemma*), $f(x)$ is analytic in $||x|| = \rho$ which contradicts to that 0 is a singular point of $f(x)$. Then we see that S consists of an element which shoes us E_0 is an one-dimensional space. This completes the proof.

§ 3. The application of the extended lemma of Schwarz.

In this chapter, we show that some of theorems[8] in the book "*Several comlex variables by S. Bochner and W. Martin*" are included in the extended lemma of Schwarz. In preceding papers, the lemma of Schwarz and the

Hadamard's three spheres theorem were extended to complex Banach spaces as follows:

The extended lemma of Schwarz[3]. *Let an E_2-valued function $f(x)$ defined in the sphere $\|x\|<R$ of E_1 be analytic and satisfy $f(0)=0$ and $\|f(x)\|\leqq M$ in the sphere $\|x\|<R$. Then*

$$\|f(x)\|\leqq \frac{M}{R}\|x\|.$$

The extended Hadamard's three spheres theorem[1]. *If $0<r_1<r_2<r_3$, $M(r_2)\leqq M(r_1)^\theta M(r_3)^{1-\theta}$, where $M(r)=\operatorname{Sup}_{\|x\|=r}\|f(x)\|$ and $\theta=\dfrac{\log r_3-\log r_2}{\log r_2-\log r_1}$ and then $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$.*

Theorem 6. *Let E_i-valued function $f_i(x)$ be analytic on $\|x\|\leqq 1$ in E_0 and satisfy $f_i(0)=0$, for $1\leqq i\leqq n$. Then*

$$\Big(\sum_{i=1}^n\|f_i(x)\|^p\Big)^{\frac{1}{p}}\leqq\|x\|\operatorname*{Sup}_{\|x\|=1}\Big(\sum_{i=1}^n\|f_i(x)\|^p\Big)^{\frac{1}{p}}.$$

Proof. In the product space $E_1\times E_2\times\cdots\times E_n$, the norm of $y=(y_1, y_2, \cdots, y_n)$ is defined as follows $\|y\|=(\sum_{i=1}^n\|y_i\|^p)^{\frac{1}{p}}$, where y_i belongs to E_i, then this product space is the complex Banach spaces. Put $F(x)=(f_1(x), f_2(x), \cdots, f_n(x))$, then $F(x)$ is an $E_1\times E_2\times\cdots\times E_n$ valued function and analytic in $\|x\|\leqq 1$. Appealing to the extended lemma of Schwarz, we have $\|F(x)\|\leqq\|x\|\operatorname{Sup}_{\|x\|=1}\|F(x)\|$, when $\operatorname{Sup}_{\|x\|=1}\|F(x)\|<\infty$. On the other hand, $\|F(x)\|=(\sum_{i=1}^n\|f_i(x)\|^p)^{\frac{1}{p}}$.

Thus we have $(\sum_{i=1}^n\|f_i(x)\|^p)^{\frac{1}{p}}\leqq\|x\|\operatorname{Sup}_{\|x\|=1}(\sum_{i=1}^n\|f_i(x)\|^p)^{\frac{1}{p}}$.

Even if $\operatorname{Sup}_{\|x\|=1}\|F(x)\|=\infty$, our inequality is also held clearly.

Corollary. *If complex valued functions $f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha)$ are regular in $|\alpha|\leqq 1$ and satisfy $f_i(0)=0$ for $1\leqq i\leqq n$, then we have*

$$\Big(\sum_{i=1}^n|f_i(\alpha)|^p\Big)^{\frac{1}{p}}\leqq|\alpha|\cdot\operatorname*{Max.}_{|\alpha|=1}\Big(\sum_{i=1}^n|f_i(\alpha)|^p\Big)^{\frac{1}{p}}.$$

Proof. Let $\|\alpha\|=|\alpha|$ and E_i be a complex plane, then we have this Corollary, appealing to Theorem 6.

Theorem 7. *Let E_i-valued function $f_i(x)$ be analytic on $\|x\|\leqq 1$ in E_0 for $1\leqq i\leqq n$, then we have*

$$\operatorname*{Sup}_{\|x\|=r_2}\Big(\sum_{i=1}^n\|f_i(x)\|^p\Big)^{\frac{1}{p}}\leqq\Big\{\operatorname*{Sup}_{\|x\|=r_1}\Big(\sum_{i=1}^n\|f_i(x)\|^p\Big)^{\frac{1}{p}}\Big\}^\theta\Big\{\operatorname*{Sup}_{\|x\|=r_3}\Big(\sum_{i=1}^n\|f_i(x)\|^p\Big)^{\frac{1}{p}}\Big\}^{1-\theta},$$

when $0<r_1<r_2<r_3\leqq 1$, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$ and $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$.

Proof. Put $F(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ and $\|F(x)\|=(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}$, then $F(x)$ is an analytic function defined on $\|x\| \leqq 1$ in E_0 and takes its values in the product space $E_1 \times E_2 \times \cdots\cdots \times E_n$.
Appealing to the extended Adamard's three spheres theorem, we have

$$\operatorname*{Sup}_{\|x\|=r_2}\|F(x)\| \leqslant (\operatorname*{Sup}_{\|x\|=r_1}\|F(x)\|^\theta(\operatorname*{Sup}_{\|x\|=r_3}\|F(x)\|)^{1-\theta},$$

where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$. Since $\|F(x)\|=(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}$, we have

$$\operatorname*{Sup}_{\|x\|=r_2}(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}} \leqq \{\operatorname*{Sup}_{\|x\|=r_1}(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}\}^\theta\{\operatorname*{Sup}_{\|x\|=r_3}(\sum_{i=1}^{n}\|f_i(x)\|^p)^{\frac{1}{p}}\}^{1-\theta}.$$

We can easily have following Corollary.

Corollary. If complex valued functions $f_1(\alpha), f_2(\alpha), \cdots, f_n(\alpha)$ are regular in $|\alpha| \leqq 1$, then we have

$$\operatorname*{Max.}_{|\alpha|=r_2}(\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}} \leqslant \{\operatorname*{Max.}_{|\alpha|=r_1}(\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}}\}^\theta\{\operatorname*{Max.}_{|\alpha|=r_3}(\sum_{i=1}^{n}|f_i(\alpha)|^p)^{\frac{1}{p}}\}^{1-\theta},$$

when $0<r_1<r_2<r_3\leqslant 1$, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$.

In these theorems, we see that it is convenient to treat analytic functions in abstract spaces.

References.

1) I. Shimoda: Notes on general analysis III, Jour. of Gakugei, Tokushima Univ. Vol. IV, 1954.

2) A. E. Taylor: Analytic functions in general analysis, Ann. R. Scuola Norm. Sup. Pisa. (2) 6 (1937).

3) I. Shimoda: Notes on general analysis (II), Jour. of Gakugei, Tokushima Univ. Vol. III, 1953.

4) The order of vector valued entire functions defined in complex plane was studied by Prof. E. Hille. See E. Hille: Functional Analysis and Semi-groups.

5) See 1), Theorem 10 at page 8.

6) Max. A. Zorn: Characterization of analytic functions in Banach spaces, Annals of Math. Vol. 46 (1945). Theorem: Let $p(x)$ satisfy the following conditions, 1) it is G-differentiable on E, 2) for $|\xi|=1$, $\|p(\xi x)\|=\|p(x)\|$, 3) there exists an x_0 in E and real number M, σ, with $\sigma>0$, such that for $\|x-x_0\|\leq\sigma$ we have $\|p(x)\|\leq M$. Then $\|p(x)\|\leq M$ for $\|x\|\leq\sigma$.

7) See I. Shimoda and K. Iseki: General Analysis in abstract spaces. Jour. of the Osaka Institute of Soc. and Tec. Vol. 1, No. 1, 1949. In these papers, the generalization of Hartogs's theorem was described roughly as Prof. A. E. Taylor pointed out.

*) Of course, E_0 is not necessarily a prodtct space of E_1 and E_2, but the proof of Lemma is applicable for this case.