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NOTES ON GENERAL ANALYSIS (III): ON THE NORM OF ANALYTIC FUNCTIONS

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In classical analysis, the absolute values of functions of complex number are closely connected with their analyticity. If the absolute values of a regular function are a constant in a neighbourhood of a point, it is a constant on whole domain on which it is defined and regular. Therefore, if the absolute values of a function are a constant on whole domain, it is a constant. In complex-Banach-spaces, the norms play the part of the absolute values of complex numbers, but the norms of analytic functions do not seem to be so closely connected with their analyticity as that of the absolute values of complex numbers. That is, there exist functions whose norms are constants on their domains and yet are not constants. We shall call such a function "*an analytic function of norm-constant*". First of all, the examples of the analytic function of norm-constant are reported and then the necessary and sufficient conditions that a function should be an analytic function of norm-constant are researched, in §1. In §2, the variation of the norm of an analytic function is researched using the method of $M(r)$ in classical analysis. The variation of the extended $M(r)$ characterizes the theory of functions in complex-Banach-spaces.

§1. Analytic functions of norm-constant

Let $x_{11}, x_{12}, x_{21}, x_{22}$ be complex numbers. The set of matrixes of 2-2-types $X=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ becomes a complex-Banach-space Ω, when we define $\|X\|=$ Max $(|x_{11}|, |x_{12}|, |x_{21}|, |x_{22}|)$. Let

$$Y=f(X)=\begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}+\rho\begin{pmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

where $0<\rho<\infty$, then $f(X)$ is an Ω-valued function defined on Ω. Clearly, $f(X)$ is a continuous function. Let α be a complex variable, then

$$f(X+\alpha Y)=\rho\begin{pmatrix} 1 & x_{12} \\ x_{21} & x_{22} \end{pmatrix}+\rho\begin{pmatrix} 0 & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\alpha .$$

This shows that $f(X)$ is G-differentiable. Therefore, $f(X)$ is analytic on whole spaces. We investigate the variety of the norm of $f(X)$ in $\|X\| \leqq 1$. When $\|X\| \leqq 1$, $|x_{ij}| \leqq 1$, where $i, j = 1, 2$. Therefore, we have

$$\begin{aligned} \|f(X)\| &= \left\| \rho \begin{pmatrix} 1 & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right\| \\ &= \rho \operatorname{Max}(1, \ |x_{12}|, \ |x_{21}|, \ |x_{22}|) \\ &= \rho . \end{aligned}$$

Thus we see that the norm of $f(X)$ is a constant ρ, when $\|X\| \leqq 1$, but $f(X)$ is not a constant. Moreover, $\|f(x)\| \equiv \rho$, when $|x_{11}| < \infty$, $|x_{12}| \leqq 1$, $|x_{21}| \leqq 1$, and $|x_{22}| \leqq 1$.

Now, an another example is reported. Let N be an arbitrary positive number and put $f(X) = \left(\frac{N+x_1}{2N}, \left(\frac{N+x_2}{2N}\right)^{\frac{1}{2}}, \ldots\ldots, \left(\frac{N+x_n}{2N}\right)^{\frac{1}{n}}, \ldots\ldots \right)$, where $X = (x_1, x_2, \ldots\ldots, x_n, \ldots\ldots)$ varies in the sphere $\|X\| < N$ in complex-l_2-spaces and $f(X)$ takes its values in complex-D_ω-spaces. $\left(\frac{N+x_n}{2N}\right)^{\frac{1}{n}}$ is considered on one branch of its Riemann surfaces. Then $f(X)$ is an one-valued analytic function and $\|f(X)\| = \sup\limits_{1 \leq n} \left| \frac{N+x_n}{2N} \right|^{\frac{1}{n}} = 1$, when $\|X\| < N$ in complex-l_2-spaces.

A function defined in a bounded domain is not necessary a constant even if the norm of the function is a constant on whole domain. Now, we investigate of analytic functions whose norms are constants on whole spaces. Let E_1 and E_2 be two complex-Banach-spaces.

Theorem 1. *Let E_2-valued function $f(x)$ be an analytic function defined for all finite values in E_1. If $\|f(x)\| = 0(\|x\|^\kappa)$ as $\|x\| \to \infty$, $f(x)$ is a polynomial of degree k at most.*

Proof. Since $f(x)$ is analytic for all finite values in E_1, $f(\alpha x)$ is expressed as follows

$$f(\alpha x) = \sum_{n=0}^{\infty} h_n(x) \alpha^n,$$

for an arbitrary point x and an arbitrary complex number α, where $h_n(x)$ is a homogeneous polynomial of degree n. Let C be a circle $|\alpha| = R$ on the complex-α-plane, then we have

$$\begin{aligned} h_n(x) &= \frac{1}{2\pi i} \int_c \frac{f(\alpha x)}{\alpha^{n+1}} d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta} x)}{R^n e^{in\theta}} d\theta , \end{aligned}$$

where $\alpha = Re^{i\theta}(0 \leqq \theta \leqq 2\pi)$. Therefore, we have

$$||h_n(x)|| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{||f(Re^{i\theta}x)||}{R^n}\, d\theta$$
$$= 0\,(R^k||x||^k)/R^n\,,$$

for sufficiently larg R, because $||f(x)|| = 0\,(||x||^k)$, as $||x|| \to \infty$. R can be taken as large as we like, $||h_n(x)|| = 0$, when $n > k$. Since x is an arbitrary point of E, $h_n(x) \equiv 0$, when $n > k$. Thus we have $f(x) = \sum_{n=0}^{k} h_n(x)$, which is a polynomial of degree k at most.

Corollary*) **1.** *If $f(x)$ is analytic and satisfies $||f(x)|| \leqq M$ on whole spaces, then $f(x)$ is a constant.*

Proof. Appealing to Theorem 1 for $k = 0$, we have $f(x) \equiv f(0)$.

Corollary 2. *If $f(x)$ is analytic and $||f(x)|| \equiv c$ on whole spaces, then $f(x)$ is a constant.*

An analytic function whose norm is a constant on whole space is a constant but a function is not necessarily a constant, even if the norm of the function is a constant on a bounded domain. It seems to be a character of the function as well as the boundedness of the function that the norm of the function is a constant. Thus, the necessary and sufficient condition that the norm of the function should be a constant must be investigated. An open and connected set is called a domain.

Lemma. *If $f(x)$ is analytic on a domain D, the set of x which satisfies $||f(x)|| = c$ is a (relative) closed set in D, where c is a constant.*

Proof. Suppose that a sequence $\{x_n\}$ converges to x_0 in D and satisfies $||f(x_n)|| = c$ for $n = 1, 2, 3, \ldots\ldots$. Since $f(x)$ is continuous and satisfies

$$\Big|\, ||f(x_0)|| - ||f(x_n)|| \,\Big| \leqq ||f(x_n) - f(x_0)||\,, \quad \lim_{n\to\infty} ||f(x_n)|| = ||f(x_0)||$$

and we have $||f(x_0)|| = c$. This completes the proof.

Theorem 2. *If an analytic function $f(x)$ defined on D satisfies $||f(x)|| \leqq M$ and moreover $||f(x_0)|| = M$ for a point x_0 in D, $||f(x)|| \equiv M$. (The inverse is also true.)*

Proof. Let S be a set of point x which satisfies $||f(x)|| = M$ in D. S is not a null set by the assumption. Let x_0 be an arbitrary point of S. Since x_0 is an inner point of D, there exists a neighbourhood $V(x_0)$, which is a set of point x satisfying $||x - x_0|| < \rho$, in D. Suppose that $S \not\supset V(x_0)$, then there exists a point y in $V(x_0) \frown CS$. Since $f(x)$ is analytic in D,

$$f(x_0) = \frac{1}{2\pi i}\int_c \frac{f(x_0+\alpha(y-x_0))}{\alpha}\, d\alpha\,,$$

where C is a circle $|\alpha|=1$. Clearly $x+\alpha(y-x_0) \in V(x_0)$ when $\alpha \in C$. Let $\alpha = e^{i\theta}(0 \leqq \theta \leqq 2\pi)$, then we have

$$||f(x_0)|| \leqq \frac{1}{2\pi}\int_0^{2\pi} ||f(x_0+e^{i\theta}(y-x_0))||\, d\theta.$$

Since $f(x)$ is continuous at y and satisfies $||f(y)|| < M$, there exist a positive number ε and δ satisfying $||f(x_0+e^{i\theta}(y-x_0))|| < M-\varepsilon$ when $|\theta| \leqq \delta$.
Then we have,

$$\begin{aligned} ||f(x_0)|| &\leqq \frac{1}{2\pi}\int_{-\delta}^{+\delta} ||f(x_0+e^{i\theta}(y-x_0))||d\theta + \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} ||f(x_0+e^{i\theta}(y-x_0))||d\theta \\ &\leqq M - \frac{\varepsilon\delta}{\pi} \end{aligned}$$

contradicting to that $x_0 \in S$. Therefore, $S \supset V(x_0)$ and we see that S is an open set. If $D \neq S$, $D = S + D \frown CS$, where $D \frown CS$ is an open set, because S is a closed set by Lemma. This contradicts to that D is connected. Therefore, $D = S$, which shows that the norm of $f(x)$ is a constant M on D.

Theorem 3. (*The extended theorem of Lindelöf.*) *If an analytic function $f(x)$ defined on a bounded domain D satisfies the followig conditions*: (1) *there exists a neighbourhood $V(x)$ of x for an arbitrary positive number ε and an arbitrary point x on the boundary of D and $||f(y)|| < M+\varepsilon$ when $y \in V(x) \frown D$,* (2) *$||f(x_0)|| = M$ for a point x_0 in D, then $||f(x)|| \equiv M$ on D. Therefore, if $||f(x)||$ is not a constant, $||f(x)|| < M$ in D.*

Proof. Let y be a point in D and α be a complex variable. $f(\alpha y)$ is an analytic function of α while $\alpha y \in D$. Plainly, a set E composed of α which satisfies $\alpha y \in D$ is an open set. Now, let S be a component of 1 in E, then S is a bounded domain in α-plane. Let a boundary of S be Γ, then, if $\alpha_0 \in \Gamma$, $\alpha_0 y$ is a boundary point of D. From the assumption (1), for an arbitrary positive number ε, there exists a neighbourhood $V(\alpha_0 y)$ and $||f(x)|| < M+\varepsilon$ when $x \in V(\alpha_0 y) \frown D$. For a sufficiently small positive number δ which is smaller than the distance between 1 and Γ, $\alpha y \in V(\alpha_0 y)$, when $|\alpha-\alpha_0| \leqq \delta$ and then Γ is covered by such circles. Since Γ is a closed bounded set on α-plane, it is covered by such circles of finite numbers. Accordingly, there are some domain S_i surrounded by the arcs of finite numbers. Among S_i, let S_0 be a domain which includes 1. Since $||f(\alpha y)||$ is subharmonic about α, $||f(\alpha y)||$ takes its maximum on the boundary of S_0 and we see that $||f(\alpha y)||< M+\varepsilon$, when $\alpha \in S_0$. Since $S_0 \ni 1$, $||f(y)||<M+\varepsilon$ and then $||f(y)|| \leqq M$, because ε is an arbitrary positive number. Thus we see that $||f(x)|| \leqq M$ on the whole domain D, because y is an arbitrary point of D. From the assumption (2) and Theorem 2, $||f(x)|| \equiv M$ on D. Finally, if $||f(x)||$ is not a constant, the assumption (2) must be false, that is, $||f(x)||<M$ for all x in D. This completes the proof.

It is plain that Theorem 2 and Theorem 3 are the necessary conditions $||f(x)||$ to be a constant.

§ 2. Extended $M(r)$

Let D be a domain in E_1 being expected to include the origin without losing generality and E_2-valued function $f(x)$ be analytic in D.

Definition 1. *A positive number σ is called a radius of norm-constant of $f(x)$ (with respect to 0), if it satisfies following conditions :* (1) $||f(x)|| \equiv C$ *for all x in $||x|| \leqq \sigma$, where C is a constant,* (2) *for an arbitrary positive number ε, there exists in the sphere $||x||<\sigma+\varepsilon$ at least a point x on which $||f(x)|| \neq C$.*

The sphere of radius σ is called *the sphere of norm-constant of $f(x)$*.

Definition 2. *When x is an arbitrary point on $||x||=1$, $r(x)$ is defined as the upper limit of r such that $||f(\alpha x)||=C$ (which is a constant) when $|\alpha| \leqq r$.*

Then we have the following theorem.

Theorem 4. *Put $\operatorname*{Inf}_{||x||=1} r(x)=\sigma$, then σ is the radius of norm-constant of $f(x)$.*

Proof. We prove this theorem for $0<\sigma<\infty$, because, when $\tau=0$ or ∞, we can discuss as well. Since $||f(0)||=C$, C is independent of the variation of x on $||x||=1$. When $|\alpha| \leqq r(x)$, $||f(\alpha x)||=C$, since $||f(\alpha x)||$ is a continuous function of α. Let y be an arbitrary point in $0<||x|| \leqslant \sigma$. Put $x=\dfrac{y}{||y||}$, then $||x||=1$ and we have $||f(||y||x)||=C$, since $||y|| \leqq \sigma = \operatorname*{Inf}_{||x||=1} r(x) \leqq r(x)$ and $||f(\alpha x)||=C$ for $|\alpha| \leqq r(x)$. On the other hand, since $y=||y||x$, $||f(y)||=C$. Since y is an arbitrary point in $0<||x|| \leqslant \sigma$, $||f(x)||=C$ for all values on $0<||x|| \leqq \sigma$. Considering the fact that $||f(0)||=C$, $||f(x)|| \equiv C$ for $||x|| \leqq \sigma$. From the definition of σ, we see that there exists at least a point x_0, which satisfies $||f(x_0)|| \neq C$, in $||x||<\sigma+\varepsilon$ for an arbitrary positive number ε. Therefore, σ is a radius of norm-constant of $f(x)$.

Theorem 5. *Let $f(x)$ be analytic in a domain D and $f(0)=0$. If $\sigma > 0$,* *then $f(x)$ is a constant* 0, *where σ is the radius of norm-constant of $f(x)$.*

Proof. From the definition of the radius of norm-constant, $\|f(x)\|=C$, when x lies in $\|x\| \leqq \sigma$, where C is a constant. Then $C=\|f(0)\|=0$. Therefore, $f(x) \equiv 0$ on $\|x\| \leqq \sigma$. Let the largest sphere including 0 in D be $\|x\|<R$. If $\sigma<R$ and y is an arbitrary point in $\|x\|<R$, $f(y)=\sum_{n=1}^{\infty} h_n(y)$. Then we have

$$h_n(y)=\frac{1}{2\pi i}\int_c \frac{f(\alpha y)}{\alpha^{n+1}}\,d\alpha\,,$$

where C is a circle such that $|\alpha|=\dfrac{\sigma}{\|y\|}$. While, $f(\alpha y)=0$ when $\|\alpha y\|=\sigma$. Then we have $h_n(y)=0$, where $n=1, 2, 3, \ldots\ldots$, and we have $f(y)=0$. This shows that $f(x) \equiv 0$ in $\|x\|<R$, since y is an arbitrary point in $\|x\|<R$. By the analytic continuation, $f(x) \equiv 0$ on D.

Corollary. *Let $f(x)$ be analytic in a domain D and put $g(x)=f(x)-f(0)$. If a radius of norm-constant of $g(x)$ is not zero, then $f(x)$ is a constant on D.*

Proof. Appealing to Theorem 5, $g(x) \equiv 0$ on D and then $f(x) \equiv f(0)$ on D.

Definition 3. *For an arbitrary point x on $\|x\|=1$, we define $M(r, x)= \operatorname{Max}_{|\alpha|=r} \|f(\alpha x)\|$, where α is a complex variable. Put $M(r)=\operatorname{Sup}_{\|x\|=r} \|f(x)\|$.*

Theorem 6. *If $f(x)$ is analytic in $\|x\|<R$ and $r<R$,* (1) *$M(r)=\operatorname{Sup}_{\|x\| \leq r} \|f(x)\|$,* (2) *$M(r)=\operatorname{Sup}_{\|x\|=1} M(r, x)$,* (3) *$M(r) \leqq M(r')$ when $r \leqq r'$, where $r'<R$.*

Proof of (1). Clearly $M(r) \leq \operatorname{Sup}_{\|x\| \leq r} \|f(x)\|$. If $M(r)<\operatorname{Sup}_{\|x\| \leq r} \|f(x)\|$, there exists in $\|x\|<r$ a point x_0 which satisfies $M(r)<\|f(x_0)\|$. $f(\alpha x_0)$ is an analytic function of α in $|\alpha|<\dfrac{R}{\|x_0\|}$ and then $\|f(\alpha x_0)\|$ is subharmonic in $|\alpha| \leqslant \dfrac{r}{\|x_0\|} (>1)$. Therefore, $\|f(\alpha x_0)\|$ takes its maximum on the boundary $|\alpha|=\dfrac{r}{\|x_0\|}$ contradicting to the fact that $\|f(x_0)\|>M(r) \geqslant \|f(\alpha x_0)\|$, where $|\alpha|=\dfrac{r}{\|x_0\|}$. Thus we see that $M(r)=\operatorname{Sup}_{\|x\| \leq r} \|f(x)\|$.

Proof of (2). Since $\|f(\alpha x)\|$ is a continuous function of α, there exists α_0 which satisfies $\|f(\alpha_0 x)\|=\operatorname{Max}_{|\alpha|=r} \|f(\alpha x)\|=M(r, x)$, $|\alpha_0|=r$. Then $M(r, x)=\|f(\alpha_0 x)\| \leqq M(r)$, for an arbitrary point x on $\|x\|=1$. Hence, $\operatorname{Sup}_{\|x\|=1} M(r, x) \leqq M(r)$. On the other hand, $\|f(y)\| \leqq M(r, x)$, when $\|y\|=r$ and $y=rx$. Hence, $\|f(y)\| \leqq \operatorname{Sup}_{\|x\|=1} M(r, x)$ for an arbitrary point y on $\|x\|=r$. Then we have $M(r) \leqq \operatorname{Sup}_{\|x\|=1} M(r, x)$. Thus we see that $M(r)=\operatorname{Sup}_{\|x\|=1} M(r, x)$.

Proof of (3). For an arbitrary positive number ε, there exists a point x_0 on $\|x\|=r$ satisfying $M(r)-\varepsilon \leqq \|f(x_0)\|$. Appealing to (1), $\|f(x_0)\| \leqq M(r')$, since $||x_0||=r<r'$. Then we have $M(r)-\varepsilon<M(r')$ for an arbitrary positive number ε. Hence, $M(r)\leqq M(r')$.

Theorem 7. *Let σ be the upper limit of r which satisfies $M(r)=||f(0)||$. Then σ is the radius of norm-constant of $f(x)$, where $f(0)\neq 0$.*

Proof. For an arbitrary positive number ε, there exists r which satisfies $\sigma-\varepsilon<r$ and $M(r)=||f(0)||$. By Theorem 6-(1), $||f(x)||\leqq M(r)$, when $||x||\leqq r$. Thus we have $||f(x)||\equiv||f(0)||$ on $||x||\leqq r$, appealing to Theoren 2. Let x_0 be an arbitrary point on $||x||=\sigma$ and x' be a point being included in both spheres $||x-x_0||\leqq\varepsilon$ and $||x||\leqq r$, then $||f(x')||=||f(0)||$. Since $||f(x)||$ is continuous, $||f(0)||=\lim_{x'\to 0}||f(x')||=||f(x_0)||$. Thus we see that $||f(x)||\equiv||f(0)||$, when x lies on $||x||\leqq\sigma$. From the definition of σ, $||f(x)||\not\equiv||f(0)||$ on $||x||\leqq r$, when $r>\sigma$. This completes the proof.

Theorem 8. (*Extended Hadamard's three spheres theorem*). *If $r_1<r_2<r_3$,*

$$M(r_2)\leqq M(r_1)^{\frac{\log r_3-\log r_2}{\log r_3-\log r_1}}M(r_3)^{\frac{\log r_2-\log r_1}{\log r_3-\log r_1}}.$$

Proof. Let x be an arbitrary point on $||x||=1$ and ρ be a positive number which satisfies $r_1^\rho M(r_1, x)=r_3^\rho M(r_3, x)$. Put $F(\alpha)=\alpha^\rho f(\alpha x)$, where α is a complex variable. Since $||F(\alpha)||$ is a subharmonic function of α, we have

$$M(r_2, x)\leqq M(r_1, x)^\theta M(r_3, x)^{1-\theta},$$

as well as the case of complex functions, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$ and $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$. By appealing to Theorem 6-(2), we have

$$\begin{aligned}M(r_2)&=\operatorname*{Sup}_{||x||=1}M(r_2, x)\leqq\operatorname*{Sup}_{||x||=1}(M(r_1, x)^\theta M(r_3, x)^{1-\theta})\\&\leqq\operatorname*{Sup}_{||x||=1}M(r_1, x)^\theta\operatorname*{Sup}_{||x||=1}M(r_3, x)^{1-\theta}\\&=(\operatorname*{Sup}_{||x||=1}M(r_1, x))^\theta(\operatorname*{Sup}_{||x||=1}M(r_3, x))^{1-\theta}\\&=M(r_1)^\theta M(r_3)^{1-\theta},\end{aligned}$$

since t^θ and $t^{1-\theta}$ is continuous when $t>0$. This completes the proof.

Theorem 9. *If $r_1<r_2$ and $M(r_1)=M(r_2)$, $f(x)$ is the function of norm-constant on $||x||\leqq r_2$. Therefore, if $f(x)$ is not a function of norm-constant in $||x||\leqq r_2$, $M(r_1)<M(r_2)$, when $r_1<r_2$.*

Proof. If $0<r<r_1$, $M(r_1)\leqq M(r)^\theta M(r_2)^{1-\theta}$ by Theorem 8, where $\theta=\dfrac{\log r_2-\log r_1}{\log r_2-\log r}$ and $1-\theta=\dfrac{\log r_1-\log r}{\log r_2-\log r}$. Since $M(r_1)=M(r_2)$, which is not

less than $\|f(0)\|$ by Theorem 6-(1), $M(r_1) \leqq M(r)^{\theta} M(r_1)^{1-\theta}$, then $M(r_1)^{\theta} \leqq M(r)^{\theta}$. Since $0<\theta<1$, $M(r_1) \leqq M(r)$. On the other hand, $M(r) \leqq M(r_1)$, by Theorem 6-(3). Then we have $M(r)=M(r_1)=M(r_2)$. Since $f(x)$ is continuous, for an arbitrary positive number ε, there exists a positive number δ such that $\|f(x)-f(0)\|<\varepsilon$, when $\|x\|<\delta$. If $r<\delta$, $M(r)=\sup_{\|x\|=r}\|f(x)\| \leqq \|f(0)\|+\varepsilon$. Then we have, $M(r_2) \leqq \|f(0)\|+\varepsilon$. Since ε is an arbitrary positive number, $M(r_2)=\|f(0)\|$. Appealing to Theorem 2, $f(x)$ is the function of norm-constant on $\|x\| \leqq r_2$. Therefore, if $f(x)$ is not a function of norm-constant on $\|x\| \leqq r_2$, $M(r_1)<M(r_2)$.

Corollary. *If $f(x)$ is analytic in $\|x\|<R$ and the norm of $f(x)$ is a constant, when $0<r<\|x\|<R$, then $f(x)$ is a function of norm-constant on $\|x\|<R$.*

Proof. Let $r<r_1<r_2<R$, $M(r_1)=M(r_2)$. Appealing to Theorem 9, $f(x)$ is a function of norm-constant on $\|x\| \leqslant r_2$. Since r_2 can be taken as close as we like to R, $f(x)$ is a function of norm-constant on $\|x\| \angle R$.

Definition 4[1)]. *If a positive number λ satisfies the following conditions* (1) *if $0<r<\lambda$, $f(x)$ is analytic and bounded in the sphere defined by $\|x\|<r$,* (2) *if $r>\lambda$, $f(x)$ can not be analytic and bounded in the sphere defined by $\|x\|<r$, then λ is called a radius of bound of $f(x)$.*

Theorem 10. *Let λ be the upper limit of r which satisfies $M(r)<\infty$, then λ is the radius of bound of $f(x)$.*

Proof. For an arbitrary positive number r, which satisfies $r<\lambda$, $M(r)<\infty$. By Theorem 6-(1), $\|f(x)\| \leqq M(r)$, when $\|x\| \leqq r$. That is, $f(x)$ is bounded on $\|x\|<r$. By the definition of λ, it is clear that there does not exist M which satisfies $\|f(x)\| \leqq M$ when $\|x\|<r$, where $r>\lambda$. Thus we see that λ is the radius of bound of $f(x)$.

Theorem 11. *If the radius of norm-constant is infinite, $f(x)$ is a constant.*

Proof. Since $\|f(x)\|$ is a constant on whole space, $f(x)$ is a constant by Corollary 2 of Theorem 1.

Thus we see that $M(r)$ varies generally as follows

$M(r)=\|f(0)\|$, when $0 \leqq r \leqq \sigma$,

$=$ steadily increases with r in the stricter sense, when $\sigma<r<\lambda$ and tends to $+\infty$, as r tends to λ,

$=+\infty$, when $\lambda \leqq r<\tau$, where τ is the radius of analyticity[1)-2)] of $f(x)$.

The following example will practically show the above stated variety of $M(r)$. Let Ω and Ω' be complex-l_2-spaces and complex-D_ω-spaces respectively. $X=(x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\ldots)$ is a point of Ω, where x_i is a complex variable ($i=1, 2, 3, \ldots\ldots$). Put Ω'-valued functions $h_n(X)$ defined on Ω as follows

$$\begin{aligned}
h_0(X) &= \left(\frac{1}{2},\ 0,\ 0,\ \ldots\ldots\ldots\ldots\right)\\
h_1(X) &= (0,\ x_1,\ x_2,\ x_3,\ \ldots\ldots)\\
h_2(X) &= (0,\ x_2^2,\ x_3^2,\ x_4^2,\ \ldots\ldots)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
h_n(X) &= (0,\ x_n^n,\ x_{n+1}^n,\ x_{n+2}^n,\ \ldots\ldots)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

then $f(X)=\sum_{n=0}^{\infty} h_n(X)$ is analytic on whole space and has $\sigma=\frac{1}{2}$, $\lambda=1$ and $\tau=+\infty$.

First of all, we must show that $h_n(X)$ is a homogeneous polynomial of degree n. Since $h_n(X)\in\Omega'$, $||h_n(X)||=\operatorname{Sup}_{0\leq i<\infty}|x_{n+i}|^n$, for $n=1, 2, 3, \ldots\ldots$, where $||X||=\sqrt{\sum_{n=1}^{\infty}|x_n|^2}$, since $X\in\Omega$. It is clear that $h_n(X)$ is a continuous function by the definition of the norm. Since

$$h_n(X+\alpha Y)=(0,\ x_n^n,\ x_{n+1}^n,\ \ldots\ldots)+n\alpha(0,\ x_n^{n-1}y_n,\ x_{n+1}^{n-1}y_{n+1},\ \ldots\ldots)+\ldots\ldots,$$

$h_n(X+\alpha Y)$ is an analytic function of α.

$$\begin{aligned}
h_n(\alpha X) &= (0,\ (\alpha x_n)^n,\ (\alpha x_{n+1})^n,\ \ldots\ldots)\\
&= \alpha^n(0,\ x_n^n,\ x_{n+1}^n,\ \ldots\ldots\ldots)\\
&= \alpha^n h_n(X).
\end{aligned}$$

This shows that $h_n(X)$ is a homogeneous polynomial of degree n.

(1) **Proof of $\tau=+\infty$.**

$$\begin{aligned}
\tau &= 1/\operatorname{Sup}_{||x||=1}\varlimsup_{n\to\infty}\sqrt[n]{||h_n(X)||}^{\,1)-2)}\\
&= 1/\operatorname{Sup}_{||x||=1}\varlimsup_{n\to\infty}\sqrt[n]{\operatorname{Sup}_{0\leq i}|x_{n+i}|^n}\\
&= 1/\operatorname{Sup}_{||x||=1}\varlimsup_{n\to\infty}\operatorname{Sup}_{0\leq i}|x_{n+i}|\\
&= +\infty,\ \text{since } |x_n|\to 0, \text{ when } n\to+\infty.
\end{aligned}$$

(2) **Proof of $\lambda=1$.**

$$\lambda=1/\varlimsup_{n\to\infty}\sqrt[n]{\operatorname{Sup}_{||x||=1}||h_n(X)||}^{\,1)}=1/\varlimsup_{n\to\infty}\sqrt[n]{\operatorname{Sup}_{||x||=1,\,0\leq i}|x_{n+i}|^n}=1.$$

(3) **Proof of $\sigma=\frac{1}{2}$.**

$||f(x)||=\operatorname{Sup}\left(\frac{1}{2},\ |\sum_{n=1}^{\infty}x_{n+i}^n|,\ i\geqq 0\right)$. When $||X||\leqq\frac{1}{2}$, $\sum_{n=1}^{\infty}|x_n|^2\leqq\frac{1}{4}$, then $|x_n|\leqq\frac{1}{2}$ for $n\geqq 1$. Thus we have $\sum_{n=2}^{\infty}|x_{n+i}|^n\leqq\sum_{n=2}^{\infty}|x_{n+i}|^2$. On the other hand, $\sum_{n=1}^{\infty}|x_{n+i}|^2 \leqq \frac{1}{4}$,

then $$|x_{1+i}| \leqq \sqrt{\frac{1}{4}-\sum_{n=2}^{\infty}|x_{n+i}|^2}.$$

Then $$|\sum_{n=1}^{\infty} x_{n+i}^n| \leqq |x_{1+i}| + \sum_{n=2}^{\infty}|x_{n+i}|^n \leqq \sqrt{\frac{1}{4}-\sum_{n=2}^{\infty}|x_{n+i}|^2}+\sum_{n=2}^{\infty}|x_{n+i}|^n \leqq \frac{1}{2}.^{3)}$$

Thus we have, $||f(X)|| = \frac{1}{2}$, when $||X|| \leqq \frac{1}{2}$. For an arbitrary positive number ε, put $X=\left(\frac{1}{2}+\varepsilon, 0, 0, \ldots\ldots\right)$, then $||f(X)||=\mathrm{Sup}\left(\frac{1}{2}, \frac{1}{2}+\varepsilon\right)=\frac{1}{2}+\varepsilon$. This shows that $\sigma=\frac{1}{2}$.

References.

1) I. Shimoda; On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

2) E. Hille; Functional Analysis and Semi-groups.

3) Put $\sum_{n=2}^{\infty}|x_{n+i}|^2=x$, $\sum_{n=2}^{\infty}|x_{n+i}|^n=y$, then $\sqrt{\frac{1}{4}-x}+y \leqq \frac{1}{2}$ may be proved easily for $0 \leqq y \leqq x \leqq \frac{1}{4}$.

*) See A. E. Taylor: Analytic function in general Analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. Vol. VI (1937), page 15, where Liouville's theorem was proved.

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ON FINITE ONE-IDEMPOTENT SEMIGROUPS (1)

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Since a finite semigroup contains at least one idempotent [1], [2], the number or the behaviour of idempotents plays an important part in the theory of finite semigroups. As the simplest case, we argue in this paper the structure of a finite one-idempotent semigroup, by which we mean a finite semigroup having only one idempotent. However the precise discussions are here limited to some special cases, and so the others will be called to account in the part II (which is unpublished). In § 1 we relate to the group that is called "Kerngruppe", and in § 2 give the condition for a one-idempotent semigroup to be a group. Especially we investigate the enclosed extension of a group in § 3, the zero-semigroup in § 4, and the power-semigroup in § 5.

§ 1. Greatest group.

Let S be a finite semigroup whose only one idempotent is e.

Lemma 1. *If a finite one-idempotent semigroup S satisfies $Se=S$ (or $eS=S$), then S is a group.*

Proof. Our proof is only restricted to the case that $Se=S$, the other being analogous. For all $x\in Se$, when x is represented by $x=ye$, $y\in S$, we get $xe=(ye)e=y(ee)=ye=x$. Hence e is a right-identity of Se. Since xS is a subsemigroup of S, it contains this e, in other words, there exists $z\in S$ such that $xz=e$ for any $x\in S$. Thus S has been proved to be a group.

Theorem 1. *Let S be a finite one-idempotent semigroup and let $G=Se$. Then the subset G has the following properties:*

(1) *G is the greatest group*[1] *in S, and $G=eS$.*

(2) *G is the least ideal of S.*

(3) *e commutes with every $x\in S$.*

(4) *S is homomorphic on G.*

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1) By the greatest group G in S we mean the group G such that $G\supseteq G_1$ for all group G_1 contained in S. The least ideal is dually defined.

Proof. (1) G is a subsemigroup whose idempotent is no other than e, and satisfies $Ge=G$. It follows from Lemma 1 that G is a group. Since e is the identity of the group G, we have $Se=G=Ge=eG\subset eS$, similarly $eS\subset Se$; and so $Se=eS$. Next, let G_1 be any group contained in S. The element e is at the same time an identity of G_1, and we have $G_1=G_1e\subset Se=G$, showing that G is the greatest group in S.

(2) Immediately it follows that

$$SG=S(Se)=(SS)e\subset Se=G\,,\ \text{and similarly}\ \ GS\subset G\,.$$

Hence G is an ideal. Let Q be any ideal of S. Then $e\in Q$, for Q is a subsemigroup of S. Therefore we have $G=Se\subset SQ\subset Q$. This shows that G is the least ideal of S.

(3) By dint of the fact that e commutes with every element xe of G, we get

$$xe=x(ee)=(xe)e=e(xe)=(ex)e=e(ex)=(ee)x=ex\,.$$

(4) The mapping φ of S onto $G=Se$ is defined as $\varphi(x)=xe$. This φ is proved to be a homomorphism by the formula:

$$\varphi(x)\varphi(y)=(xe)(ye)=x(ey)e=x(ye)e=(xy)(ee)=(xy)e=\varphi(xy)$$

where $x,y\in S$ and $ey=ye$ because of the above (3). q. e. d.

In particular if e is a right (left) identity, S is a group by Lemma 1. If e is a right (left) zero, e is a two-sided zero. Then, however, Theorem 1 becomes trivial, whence we must investigate this case on different standpoints (cf. § 4).

All one-idempotent semigroups of order[2] n are classified into n classes according to the order $k\,(k=1,2,\cdots,n)$ of the greatest group in itself. We call the order n the *d-order n of* S, and call the order k the *g-order of* S, or we say that S is of g-order k under d-order n. Especially when S is of g-order 1, S is called a zero-semigroup[3] or z-semigroup.

§ 2. *v*-order and the condition for a group.

The subset SS or S^2 is called the value-range of the semigroup S. When the order of the value-range is m, we say that S is of *v-order m*. This v-order plays a remarkable rôle in our theory as much as the g-order does.

2) By "order of S" we mean the number of elements of S.

3) "Zero-semigroup" defined here is more general than what was done by Rees.

Among the above three kinds of orders, n, m, k, obviously holds $k \leqq m \leqq n$. If the v-order of S equals to its d-order, S is said to be universal.

Now, with respect to universal one-idempotent semigroups, the following Theorem 2 is worthy of our notice. In the below Lemma 2, S is not necessarily one-idempotent.

Lemma 2. *Let S be a finite universal semigroup with a zero. If S has a non-void subset X such that $X \subset SX$ and $X \frown S(S-X) = \phi$, then S contains an idempotent different from zero.*

Proof. Let n be the d-order of S and l the order of X. We shall prove the lemma by induction with respect to l. At first when $l=1$, setting $X=\{p\}$, $p \neq 0$[4], we have $p=xp$ for some $x \in S$. Let us denote by A the set of above x for fixed p, then $0 \overline{\in} A$ and A forms a subsemigroup because $(xy)p = x(yp) = xp = p$ for $x, y \in A$; and so A contains an idempotent, consequently S has an idemptotent, which is different from zero.

Next, we shall prove the case for l under the assumption that the lemma holds for $i \leqq l-1$. Take any $a \in X$ such that $Sa \frown X \neq \phi$, then we may assume $Sa \not\supset X$ without losing generality. For, if there is an element a which satisfies $X \subset Sa$, $a \in X$, it follows that $a = xa$ for some $x \in S$; and then we can apply again the method used in the case when $l=1$.

Now, let $X_1 = X - Sa \frown X (\neq \phi)$, we get $S(S-X_1) = S(S-X) \smile Sa$, and $X_1 \frown S(S-X) \subset X \frown S(S-X) = \phi$, while $X_1 \frown Sa = \phi$. Hence $X_1 \frown S(S-X_1) = \phi$. Furthermore, since S is universal, $X_1 \subset SS = S(S-X_1) \smile SX_1$, so that $X_1 \subset SX_1$. X_1 being of order less than l, the supposition of induction enables us to conclude that S has an idempotent which is not zero. This complets the proof.

Corollary. *If S is a finite zero-semigroup, then S is non-universal.*

Proof. Suppose that S is universal. Setting $X = S - \{0\}$, X satisfies the condition of Lemma 2. Therefore S contains an idempotent different from zero, contradicting with the assumption.

Theorem 2. *Let S be a fimte one-idempotent semigroup. If S is universal, then S is a group.*

Proof. Let us suppose that S is not a group. Then, by Theorem 1, S has the greatest group $G(\neq S)$ which is an ideal at the same time, that is, $k < n$[5]

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4) The condition $X \frown S(S-X) = \phi$ leads to $p \neq 0$.

5) We denote by n the d-order, by k the g-order of S respectively.

according as notations in the end of §1. Denote by S^* the difference semigroup of S modulo G, which is due to Rees [2]. Clearly S^* is a zero-semigroup of order i, $2 \leqq i < n$. Now, let a^* be an arbitrary non-zero element of S^*.

Then the inverse image $a(\in S)$ of a^* is represented as $a = bc$[6], where $b \in S-G$, $c \in S-G$, for S is universal. Hence $a^* = b^*c^*$ where $b^* \in S^*$, $c^* \in S^*$; it follows that S^* is universal. This conflicts with the Corollary.

From Theorem 2, we conclude that if a semigroup S has g-order less than d-order, S is non-universal.

§ 3. Enclosed extension of a group.

If the value-range S^2 of S coincides with its greatest group G and S is not a group, S is called an enclosed extension of the group G. This paragraph is an attempt to clear its structure. Now let us denote by $a_i (i = 1, \cdots, k)$ all elements of G and by p_i the number of elements $x \in S$ which are mapped into $a_i \in G$ by the homomorphism φ which is introduced in the proof of (4) of Theorem 1. Of course $p_i \geqq 1$. Then an enclosed extension $S = \{a_1, \cdots, a_n\}$ of the group $G = \{a_1, \cdots, a_k\}$, $k < n$, is associated with an ordered system of positive integers $(p_1, \cdots, p_k)$ where $\sum_{i=1}^{k} p_i = n$.

The system $(p_1, \cdots, p_k)$ is termed the character of S, because the system determines S as the following theorems indicate.

Theorem 3. *If there are given a group $G = \{a_1, \cdots, a_k\}$, a set $S = \{a_1, \cdots, a_k, \cdots, a_n\}$ which contains G, and an ordered system of positive integers $(p_1, \cdots, p_k)$ where $\sum_{i=1}^{k} p_i = n$, $p_i \geqq 1$, then an enclosed extension S of G admitting $(p_1, \cdots, p_k)$ to be its character is determined uniquely except for isomorphism.*

Proof. At first we assign a mapping $\varphi : a_i \xrightarrow{\varphi} \varphi(a_i) = ea_i = a_i e$[7] $(i = 1, \cdots, n)$ such that φ has $(p_1, \cdots, p_k)$ as its character, in other words, the number of elements $x \in S$ which fulfil $\varphi(x) = a_i$ is p_i $(i = 1, \cdots, k)$. We can then prove that the extension S is uniquely constructed under the above φ. In fact, let us define the product xy in S as $xy = \varphi(x)\varphi(y)$[8]. Then, since φ maps an element of G into itself, we have $\varphi(xy) = \varphi(x)\varphi(y)$ which proves that φ is a homomorphism. Conversely it is obvious that the above definition of xy in S is necessary under the assumption that φ is a homomorphism.

Next, we shall prove the uniqueness of S except for isomorphism. Let,

6) Of course $b \neq 0$, $c \neq 0$, so that $b^* \neq 0^*$, $c^* \neq 0^*$.

7) e is an idempotent of S, i.e., the unit of G; and of course $\varphi(a_i) \in G$.

8) The product in the right side is that in G.

now, φ' be another mapping (of S on G) which has a character in common with φ; and then it is sufficient to show that two semigroups obtained by φ and φ' are isomorphic each other. Now the common character of φ and φ' enables us to give a permutation[9] ψ of S on itself associating x with one of elements x', which satisfy

(1) $x' = x$ for $x \in G$,

(2) $\varphi'(x') = \varphi(x)$ for $x \in S$.

Then we get, for every $x, y \in S$,

$$\begin{aligned}
\psi(x)\psi(y) &= \varphi'(\psi(x)\psi(y)) && (\because\ \psi(x)\psi(y) \in G, \text{ and } \varphi'(z) = z \text{ for } z \in G)\\
&= \varphi'(\psi(x))\varphi'(\psi(y)) && (\text{by homomorphism})\\
&= \varphi(x)\varphi(y) && (\text{by } (2))\\
&= \varphi(xy) && (\text{by homomorphism})\\
&= xy && (\because\ \varphi(z) = z \text{ for } z \in G)\\
&= \psi(xy). && (\text{by } (1))
\end{aligned}$$

Therefore ψ is an isomomorphism. Thus the theorem has been proved.

Now, G and n given, under what condition is an extension obtained by $(p_1, \cdots, p_k)$ isomorphic with that obtained by $(q_1, \cdots, q_k)$? Generally, when $(p_{i_1}, \cdots, p_{i_k})$ is got by permutating $(p_1, \cdots, p_k)$, we denote it by

$$(p_{i_1}, \cdots, p_{i_k}) = \sigma(p_1, \cdots, p_k) \qquad \text{where } \sigma = \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix};$$

that is, we say that $(p_1, \cdots, p_k)$ is permutated to $(p_{i_1}, \cdots, p_{i_k})$ by σ. Similarly, if we apply σ to G, we mean a permutation in G written

$$\sigma\{a_1, \cdots, a_k\} = \{a_{i_1}, \cdots, a_{i_k}\} \quad \text{where } a_j \xrightarrow{\sigma} a_{i_j},$$

or we say that $(a_1, \cdots, a_k)$ is permutated to $(a_{i_1}, \cdots, a_{i_k})$.

Theorem 4. *There are given the set $S = \{a_1, \cdots, a_n\}$ and the group $G = \{a_1, \cdots, a_k\}$. Let us denote by S_p an enclosed extension of G with a character $(p_1, \cdots, p_k)$, by S_q one with $(q_1, \cdots, q_k)$, and suppose that a_1 is an idempotent in each of S_p and S_q. In order that S_p is isomorphic with S_q, it is necessary and sufficient that the following conditions are satisfied.*

(1) $p_1 = q_1$.

(2) *There exists a permutation σ such that*

a) *$(p_1, \cdots, p_k)$ is permutated to $(q_1, \cdots, q_k)$ by σ, and*

b) *the mapping $\{a_1, \cdots, a_k\} \to \sigma\{a_1, \cdots, a_k\}$ is an automorphism in the group G.*

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9) A permutation is a one to one mapping of S onto itself.

Proof. At first we shall prove the necessity of the theorem. Let us suppose that S_p is mapped isomorphically on S_q by η. Since the greatest group of S_p is mapped on that of S_q by η, the contraction σ of η to G is an automorphism of G. It is obvious that $(p_1, \cdots, p_k)$ is permutated to $(q_1, \cdots, q_k)$ because of the isomorphism η. Now by φ_p and φ_q we mean homomorphisms of S_p and S_q on G respectively. Then we have $p_1 = q_1$, because the idempotent of S_p is mapped to the idempotent of S_q and $\varphi_p(x) = a_1$ implies $\varphi_q(\eta(x)) = a_1$.

Next, we shall prove the sufficiency. Suppose that $(p_1, p_2, \cdots, p_k)$ and $(p_1, q_2, \cdots, q_k)$ satisfy the conditions (1) and (2) in this theorem. Then we can define a permutation η of S_p on S_q i.e., $\{a_{j(1)}, \cdots, a_{j(n)}\} = \eta\{a_1, \cdots, a_n\}$ with the properties as follow:

(1) the contraction η to G coincides with σ,

(2) $j(i) = i$ for $i=1, k+1, k+2, \cdots, n$.

(3) $\eta(ex) = e \cdot x$ where $e \cdot x$ is the product in S_q.

Considering that $xy = (ex)(ey)$ for $x, y \in S_p$, $x \cdot y = (e \cdot x) \cdot (e \cdot y)$ for $x, y \in S_q$, and $\eta(x) = x$ for $x \in S - G$, we have

$$\eta(xy) = \eta((ex)(ey)) = \eta(ex)\eta(ey) = (e \cdot x) \cdot (e \cdot y) = x \cdot y = \eta(x) \cdot \eta(y) \text{ for } x, y \in S - G,$$
$$\eta(bx) = \eta(b(ex)) = \eta(b) \cdot \eta(ex) = \eta(b) \cdot (e \cdot x) = \eta(b) \cdot x \text{ for } b \in G,\ x \in S - G.$$

Similarly $\eta(xb) = x \cdot \eta(b)$ for $b \in G$, $x \in S - G$.

Hence η proves to be an isomorphism of S_p on S_q. Thus the proof of the theorem has been completed.

§ 4. Zero-semigroups.

In this paragraph we shall relate to some remarkable properties of a zero-semigroup S. Let us denote by 0 the zero of S. We mean by an annihilator of S an element $a \in S$ which satisfies $ax = xa = 0$ for all $x \in S$. Of course a two-sided zero is an annihilator. The following theorem is of significance for the study of its structure.

Theorem 5. *A finite zero-semigroup S of d-order no less than 2 has at least one annihilator except the zero.*

Proof. The theorem holds if the d-order is 2, since a zero-semigroup of d-order 2 is no other than

$$\begin{array}{c|cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & 0 \end{array}\ .$$

We shall prove the theorem for n under the assumption of validity for $n-1$. By the corollary in § 2, S has a subset[10] S_{n-1} such that $S^2 \subset S_{n-1}$, for which we set $S = S_{n-1} \cup \{p\}$, $p \bar{\in} S_{n-1}$. Since S_{n-1} is clearly a zero-semigroup, we can find a non-zero annihilator b of S_{n-1}. Here we may assume that either $bp \neq 0$ or $pb \neq 0$. For if $bp = pb = 0$, b becomes an annihilator of S, solving the present problem. Now, let us consider the element $c = pbp$ in S, for which the two cases may be considered: $c = 0$, $c \neq 0$.

(1) When $c = 0$ with $bp \neq 0$, it will be proved that bp is an annihilator of S.

In fact, clearly $p(bp) = 0$ and $z(bp) = (zb)p = 0p = 0$ for $z \in S_{n-1}$,
while $(bp)z = b(pz) = 0$ for $z \in S_{n-1}$ because $pz \in S_{n-1}$.

It follows that $bp (\neq 0)$ is an annihilator of S.

(2) When $c \neq 0$, c itself is one required.
Because, for every $z \in S$,

$$(pbp)z = p\{b(pz)\} = p0 = 0, \text{ and } z(pbp) = \{(zp)b\}\, p = 0p = 0$$

Since $pz,\ zp \in S_{n-1}$. Thus we completes the proof.

Theorem 6. *Let S be a zero-semigroup of d-order n. We can put all elements of S in order*

$$a_1, a_2, \cdots, a_n,$$

such that, setting $S_i = \{a_1, \cdots, a_i\}$, it holds that

$$SS_i \subset S_{i-1} \text{ and } S_iS \subset S_{i-1} \text{ for } i = 2, 3, \cdots, n.$$

Proof. Let a_1 be the zero and a_2 be an annihilator different from zero. (see Theorem 5.) Obviously $S_2S \subset S_1 \subset S_2$, $SS_2 \subset S_1 \subset S_2$; S_2 is an ideal of S. Now, after the ideal $S_{i-1} = \{a_1, \cdots, a_{i-1}\}$ (for $i \geqq 3$) is constructed such that $S_{i-1}S \subset S_{i-2}$ and $SS_{i-1} \subset S_{i-2}$, we obtain S_i by adding a_i whose image $a_i^{(i-1)}$ into the difference semigroup[11] $S^{(i-1)} = (S : S_{i-1})$ of S modulo S_{i-1} is a non-zero annihilator, that is, $a_i^{(i-1)}S^{(i-1)} \subset \{0^{(i-1)}\}$ and $S^{(i-1)}a_i^{(i-1)} \subset \{0^{(i-1)}\}$. Consequently we have $a_iS \subset S_{i-1}$, $Sa_i \subset S_{i-1}$. Then it follows immediately

$$S_iS = S_{i-1}S \cup a_iS \subset S_{i-2} \cup S_{i-1} = S_{i-1},$$

similarly $SS_i \subset S_{i-1}$. The proof of the theorem has been accomplished.

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10) S_{n-1} is composed of $n-1$ elements.

11) Rees denoted by $S^* = S - G$ the difference semigroup S^* of S modulo G, but we denote it by $S^* = (S : G)$.

By Theorem 6, one part of the structure of one-idempotent semigroups is clarified.

Theorem 6′. *Let S be a one-idempotent semigroup of d-order n. We can put all elements of S in order*

$$a_1, a_2, \cdots, a_k, a_{k+1}, \cdots, a_n$$

such that

(1) *$G=\{a_1, \cdots, a_k\}$ is the greatest group of S.*

(2) *Setting $S_i=\{a_1, \cdots, a_i\}$, it holds that*

$$SS_i \subset S_{i-1} \text{ and } S_iS \subset S_{i-1} \text{ for } i=k+1, k+2, \cdots, n.$$

§ 5. Power semigroups.

Finally we take up here finite power semigroups as the simplest example of one-idempotent semigroups. Generally S is called a power semigroup if it is generated by only one element a of S, that is, S is composed of powers of a:

$$S=\{a, a^2, a^3, \cdots, a^n \cdots\}.$$

In particular, if S is finite, then some elements appear infinitely many times in this power sequence.

Let n be the d-order of S, λ_0 be the minimum of positive integers λ which have λ' greater than λ such that $a^\lambda = a^{\lambda'}$, and μ_0 be the minimum of positive integers μ which satisfy $a^{\lambda_0}=a^\mu$ where $\mu > \lambda_0$[12]. Then we see that $\mu_0=n+1$. Poole termed this λ_0 "the order", but here we call it the "origin" of S not to confuse with the orders already defined.

It follows that $a^\lambda \neq a^{2\lambda}$ for $\lambda < \lambda_0$ and $a^\lambda = a^\kappa$ for $\lambda, \kappa \geqq \lambda_0$ if and only if $\lambda \equiv \kappa$ (mod. $\mu_0-\lambda_0$). Then the following theorem is easily proved.

Theorem 7. *A finite power semigroup S is one-idempotent and commutative, and the set*

$$a^{\lambda_0}, a^{\lambda_0+1}, \cdots, a^{\mu_0-1}$$

forms the greatest group of S, which is cyclic.

Now, let S and S' be two power semigroups with d-orders n, n', and with the origins λ_0, λ_0' respectively. Clearly S is isomorphic with S' if and only if $n=n'$ and $\lambda_0=\lambda_0'$. While the d-order n is fixed, only the origin determines the types of power semigroups.

Theorem 8. *The d-order and the origin characterize a finite power semi-*

12) See [1] or [2].

group. It follows from this that we have n types of non-isomorphic power semigroups of d-order n.

In the end of this paper, we shall give the following theorem referring to power zero semigroups.

Theorem 9. *If S is a zero-semigroup with d-order n and with v-order $n-1$, then S is a power semigroup.*

Proof. Theorem 6 makes it possible to put all elements of S in order[13)]

$$a_1, a_2, \cdots, a_n$$

such that $SS_i \subset S_{i-1}$ and $S_iS \subset S_{i-1}$ where $S_i = \{a_1, \cdots, a_i\}$.

We may assume that $S = S_n$, $S^2 = S_{n-1}$. Now we shall prove that

$$a_i = a_{i+1}a_n$$

under the supposition $a_\lambda = a_{\lambda+1}a_n$ ($\lambda = i+1, i+2, \cdots, n-1$).

By $SS_i \subset S_{i-1}$, and $S_iS \subset S_{i-1}$, we get easily $a_i \in (S-S_i)^2$, in other words

$$a_i \in (S-S_i)a_{i+1} \cup (S-S_i)a_{i+2} \cup \cdots \cup (S-S_i)a_n$$

or

$$a_i \in (S-S_i)a_n.^{14)}$$

From the assumption of induction, it follows that

$$a_i = a_{i+1}a_n.$$

On the other hand, by $SS_{n-1} \subset S_{n-2}$ and $S_{n-1}S \subset S_{n-2}$, we get

$$a_{n-1} \in SS-(SS_{n-1} \cup S_{n-1}S), \text{ that is, } a_{n-1} = a_n^2.$$

Thus every element of S is proved to be represented as a power of a_n.

Addendum.

As the corollaries of Theorem 6, 6′, we add the following two.

Corollary. *A zero-semigroup S of d-order n ($n > 2$) is homomorphic on a zero-semigroup of d-order $n-i$, $1 \leqq i \leqq n-1$.*

Proof. S_{i+1} mentioned in Theorem 6 is an ideal. It is sufficient to consider the difference semigroups of S modulo S_{i+1}.

Corollary. *A one-idempotent semigroup S of d-order n, and of g-order k is homomorphic on a zero-semigroup of d-order $n-i$, $k-1 \leqq i \leqq n-1$.*

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13) Of course, a_1 is a zero, and a_2 is an annihilator distinct from zero.

14) For, $a_\lambda = a_{\lambda+1}a_n$ ($\lambda = i+1, i+2, \cdots, n-1$).

The proofs of some theorems in this paper have been much improved in my paper to appear: "On compact one-idempotent semigroups."

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EINE VERALLGEMEINERUNG DES ABELSCHEN LIMITIERUNGSVERFAHREN BEI INTEGRALEN

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Das Abelsche Mittel für das Integral $s(x)=\int_a^x f(u)du$ wird gewöhnlich durch

$$(1)\qquad \lim_{\lambda\to+0}\int_a^\infty e^{-\lambda u}f(u)\,du=\lim_{\lambda\to+0}\lambda\int_a^\infty e^{-\lambda u}s(u)\,du\equiv\lim_{\lambda\to 0,\ u\to\infty}A(\lambda,u)$$

definiert. Es wird aber sich in gewißem Falle als zweckmäßig erweisen, dafür ein verallgemeinertes Mittel

$$(2)\qquad \lim_{\lambda\to 0,\ u\to\infty}A(\lambda,g(u))\equiv\lim_{\lambda\to+0}\int_a^\infty \exp\{-\lambda g(u)\}\,f(u)\,du$$

$$=\lim_{\lambda\to+0}\lambda\int_a^\infty \exp\{-\lambda g(u)\}\,g'(u)\,s(u)\,du$$

zu annehmen, wobei $g(u)\,(=v)$ irgendeine monoton zunehmende reelle positive Funktion mit $\lim_{u\to\infty}g(u)=\infty$ ist. Dies ist offenbar eine spezielle Toeplitzsche Transformation, weil die dazu notwendigen und hinreichenden Bedingungen

$$\lim_{\lambda\to+0}\lambda\int_a^\infty \exp\{-\lambda g(u)\}\,g'(u)\,du=1,$$

$$\lambda\int_a^\infty |\exp\{-\lambda g(u)\}\,g'(u)|\,du<K\qquad \text{für}\quad 0<\lambda\leqq\lambda_0,$$

und $$\lim_{\lambda\to+0}\lambda\exp\{-\lambda g(u)\}\,g'(u)=0\qquad \text{bei festes } u,$$

sämtlich stets bestehen. Folglich erfüllt es die Permanenzbedingung, daß jedesmal $A(\lambda,g(u))\to l$ strebt, wenn $\lim_{u\to\infty}s(u)=l$ vorhanden ist. Freilich ist die Umkehrfunktion $u=g^{-1}(v)$ wieder positiv monoton zunehmend mit $\lim_{v\to\infty}g^{-1}(v)=\infty$. Wenn hiernach

$$f(u)\,du=f(g^{-1}(v))\,(g^{-1}(v))'\,dv=f_1(v)\,dv$$

und $$s(x)=\int_a^x f(u)\,du=\int_{g(a)=b}^{g(x)=y}f_1(v)\,dv=s_1(y)$$

gesetzt wird, so wird

$$\lim A\ g(u)\ s(x)=\lim_{\lambda\to+0}\int_b^\infty e^{-\lambda v}f_1(v)\,dv=\lim_{\lambda\to+0}\lambda\int_b^\infty e^{-\lambda v}s_1(v)\,dv,$$

was mit dem gewöhnlichen Abelschen Mittel (1) übereinstimmt. Trotzdem ist unser neues Mittel (2) bisweilen vielmehr noch bequemer, wie unten gezeigt wird.

§ 1. Wir wollen zuerst dartun, daß unser neuer Limes (2) tatsächlich der Erweiterungsbedingung genügt: Es ist nämlich bei Knopp[1] hingeweist worden, daß die Reihe $\sum_{n=1}^{\infty} \frac{1}{n^{1-\beta i}} (\beta \gtrless 0)$ nicht konvergiert, und folglich weder Cesàrosches noch Abelsches Mittel haben kann. Ganz ebenso verhält sich das Integral $\int_1^\infty \frac{du}{u^{1-\beta i}} (\beta \gtrless 0)$; d. h. es ist weder konvergent noch Cesàro-, noch Abel-limitierbar in gewöhnlichen Sinne.[2] Trotzdem hat das fragliche Integral wirklich ein verallgemeinertes Abelsches Mittel. In der Tat wird bei $g(u) = \log u$ mit $\arg u = 0$ unser neues Mittel

$$\int_1^\infty \exp\{-\lambda \log u\} \frac{du}{u^{1-\beta i}} = \int_1^\infty u^{-\lambda-1+\beta i}\, du = \left[\frac{u^{-\lambda+\beta i}}{-\lambda+\beta i}\right]_1^\infty \qquad (\lambda > 0),$$

damit das betreffende Integral als sein $\boldsymbol{A}$-log-Mittel den Wert i/β hat. Ebensowohl gilt

$$\int_0^1 \exp\{-\lambda |\log u|\} \frac{du}{u^{1-\beta i}} \to \frac{-i}{\beta} \qquad \text{bei} \quad \lambda \to +0,$$

und daher ist

$$\lim_{\lambda \to +0} \int_0^\infty \exp\{-\lambda |\log u|\} \frac{du}{u^{1-\beta i}} = 0.\text{[3]} \tag{3}$$

Ein noch annehmbares Beispiel ist $\int_0^\infty \exp\left(-\frac{1}{u}\right) u^{-1-\beta i}\, du$, wo $\beta \gtrless 0$ und $\arg u = 0$ ist. Da hier der Integrand $= O(u^{-1})$ für $u \to \infty$ bleibt, so ist das Integral weder konvergent, noch C_k-limitierbar für jedes $k > 0$, noch A-limitierbar in üblichem Sinne. Trotzdem ist es zwar $\boldsymbol{A}$-log-limitierbar. Denn, wird $u = 1/t$ gesetzt, so folgt für $\lambda > 0$

$$\int_0^\infty \exp\left\{-\lambda \log u - \frac{1}{u}\right\} u^{-1-\beta i}\, du = \int_0^\infty e^{-t} t^{\lambda-1+\beta i}\, dt = \Gamma(\lambda + \beta i). \tag{4}$$

Daher ist der gesuchte $\boldsymbol{A}$-log-Limes eben $\Gamma(\beta i)$.

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1) K. Knopp, Unendliche Reihen, zweite Auflage, S. 487.

2) Y. Watanabe, Das Abelsche Limitierungsverfahren bei Integralen, Japanese Journal of Mathematics, Vol. IX (1932), S. 177-197. Aber besser vergleiche man die berümte Arbeit von J. Karamata; Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjesche Transformation betreffen, Journal für reine und angewandte Mathematik, Vol. 164 (1931), S. 27-39.

3) Vgl. unten den Fall I in § 2.

Es sei nun bemerkt, daß für unseren neuen Abelschen Limes die bekannte Tauber-Karamatasche Satz (Umkehrung des Abelschen Satzes) etwas anders ausgedrückt werden soll; d. h. aus $\lim\limits_{\lambda\to 0}\int_a^\infty \exp\{-\lambda g(u)\} f(u)\,du = l$, aber nicht mit $f(u)=O\left(\frac{1}{u}\right)$, sondern unter Bedingung $f(u)=O(g'(u)/g(u))$ bei $u\to\infty$, der Schluß $\int_a^\infty f(u)\,du = l$ folgt. Ist nämlich $g(u)=v$, so wird $f(u)=f(g^{-1}(v))$ $(g^{-1}(v))'\,\frac{dv}{du}=f_1(v)g'(u)$ und demnach $f(u)=O\left(\frac{1}{v}\right)g'(u)=O\left(\frac{g'(u)}{g(u)}\right)$ sein soll.

§ **2.** Ich überlege mir als zweites Beispiel das Integral

$$\int_\varepsilon^\omega u^{\alpha-1+\beta i}\,du \tag{5}$$

wobei $\alpha \gtreqqless 0$, $\beta \gtrless 0$ mit $\arg u=0$, $\varepsilon\to 0$, $\omega\to\infty$ sind. Oder, wird $u=e^v$ gesetzt, so kommt

$$\int_{\log\varepsilon}^{\log\omega} \exp\{\alpha v+\beta v i\}\,dv, \qquad \log\varepsilon\to-\infty,\ \log\omega\to+\infty, \tag{6}$$

welches zugleich drittes Beispiel darbietet. Eine formale Integration liefert

$$\frac{1}{\alpha+\beta i}\Big[u^{\alpha+\beta i}\Big]_\varepsilon^\omega = \frac{1}{\alpha+\beta i}\Big[\exp\{(\alpha+\beta i)\log u\}\Big]_\varepsilon^\omega,$$

so daß entweder für $\omega\to\infty$ bei $\alpha>0$, oder für $\varepsilon\to+0$, bei $\alpha<0$ der absolute Betrag des Reel- bezw. Imaginär-Teil zwischen immer großer werdenden positiven und negativen Grenzen hin- und herpendelt, während bei $\alpha=0$ für $\omega\to\infty$ sowie $\varepsilon\to+0$ noch ∞ mal endlich schwingt. Auch wird das gewöhnliche Abelsche Mittel (1) bei $\alpha>0$ durch

$$\begin{aligned}\int_0^\infty e^{-\lambda u}u^{\alpha-1+\beta i}\,du &= \int_0^\infty e^{-t}t^{\alpha-1+\beta i}\,dt/\lambda^{\alpha+\beta i} \qquad (t=\lambda u)\\ &= \Gamma(\alpha+\beta i)/\lambda^{\alpha+\beta i}\end{aligned}$$

gegeben, was bei zu $+0$ abnehmenden λ wieder unendlich groß schwingend wird; Oder, wenn man auch $u=e^v$ in das Integral einsetzt und berechnet

$$\int_{-\infty}^\infty \exp\{(-\lambda+\alpha+\beta i)v\}\,dv = \left[\frac{\exp(-\lambda+\alpha+\beta i)v}{-\lambda+\alpha+\beta i}\right]_{-\infty}^\infty$$

daraus aber wird gar nichts.

Nun spielt unser neues $\boldsymbol{A}$-v^2-Mittel eine vorherrschende Rolle: Es ist nämlich

$$A(\lambda)=\int_{-\infty}^{\infty} e^{-\lambda v^2} e^{\alpha v+\beta v i}\,dv$$

$$=\exp\frac{\alpha^2+2\alpha\beta i}{4\lambda}\int_{-\infty}^{\infty}\exp\left\{-\lambda\left(v-\frac{\alpha}{2\lambda}\right)^2+i\beta\left(v-\frac{\alpha}{2\lambda}\right)\right\}dv.$$

Wenn man in dem Integral $v-\dfrac{\alpha}{2\lambda}=x$ einführt, ~~und das Integral im Cauchyschen Sinne versteht,~~*) so erhält man

$$\int_{-\infty}^{\infty} e^{-\lambda x^2}(\cos\beta x+i\sin\beta x)\,dx=\sqrt{\frac{\pi}{\lambda}}\exp\left\{-\frac{\beta^2}{4\lambda}\right\},\qquad(\lambda>0)$$

wie man leicht mit Benutzung der Cauchyschen Integration funktionentheoretischweise erweisen kann. Tatsächlich ist die komplexe Funktion $f(z)=\exp\{-\lambda z^2\}$ von $z=x+yi$ regulär in dem durch vier Seiten $y=0$,
$\lambda=R'$ $x=\pm R$ und $y=\beta i/\lambda$ geschlossenen rechteckigen Bereiche, so daß das über den Rand C des Rechteckes erstreckte Integral $\oint f(z)\,dz=0$ wird, und woraus für $R\to\infty$ (R') die obige Formel sofort folgt. Demnach gilt für $\lambda>0$

$$A(\lambda)=\sqrt{\frac{\pi}{\lambda}}\exp\frac{\alpha^2-\beta^2+2\alpha\beta i}{4\lambda},\tag{7}$$

und dies strebt gegen Null für $\lambda\to+0$, falls $\alpha^2<\beta^2$ ist. Also ist das A-v^2-Mittel des Integrales $\int_{-\infty}^{\infty}\exp(\alpha v+\beta vi)\,dv$ gleich 0; Oder wird das A-$(\log u)^2$-Mittel des Integrales $\int_0^{\infty}u^{\alpha-1+\beta i}\,du$ gleich 0, falls $|\alpha|<|\beta|$ ist.

§ **3.** Bisher hat man stillschweigend vorausgesetzt, daß der Parameter λ bloß reel positiv und gegen Null strebend sei. Aber doch es kann weiter

---

*) Wenn hier $\int_{-\infty}^{\infty}$ als $\lim_{R\to-\infty,\,R'\to\infty}\int_R^{R'}$ verstanden wird, so bringt es auf neue Rechnung:

$$A(\lambda)=\exp\frac{\alpha^2+2\alpha\beta i}{4\lambda}\int_0^{\infty}e^{-\lambda x^2}(\cos\beta x+i\sin\beta x)\,dx$$

$$\lim_{R'\to\infty}\int_0^{R'}=\exp\frac{\alpha^2+2\alpha\beta i-\beta^2}{4\lambda}\left[\frac{\sqrt{\pi}}{2\sqrt{\lambda}}+\frac{i}{\sqrt{\lambda}}\int_0^{\beta/2\sqrt{\lambda}}e^{y^2}dy\right].$$

Daher strebt bei $\alpha^2<\beta^2$ für $\lambda\to+0$ das mit dem reellen Teil im Klammern multiplizierte Glied noch gegen 0, was aber bei dem imaginären Teil nicht der Fall ist. Denn, es ist

$$\lim\exp\frac{\alpha^2+2\alpha\beta i-\beta^2}{4\lambda}\cdot\frac{1}{\sqrt{\lambda}}\int_0^{\beta/2\sqrt{\lambda}}e^{y^2}dy=\lim\exp\frac{\alpha^2+2\alpha\beta i}{4\lambda}\lim\int_0^{\beta/2\sqrt{\lambda}}e^{y^2}dy\Big/\sqrt{\lambda}\,e^{\beta^2/4\lambda},$$

worin der zweite Faktor $=\dfrac{\infty}{\infty}=\lim\limits_{\lambda\to+0}\dfrac{e^{\beta^2/4\lambda}\left(-\beta/4\lambda\sqrt{\lambda}\right)}{e^{\beta^2/4\lambda}\left(\dfrac{1}{2\sqrt{\lambda}}-\dfrac{\beta^2}{4\lambda\sqrt{\lambda}}\right)}=\dfrac{1}{\beta}$

strebt, während der erste Faktor $\exp\dfrac{\alpha^2+2\alpha\beta i}{4\lambda}$ für $\lambda\to+0$, noch unendlich schwingend wird, sofern $\alpha\beta\neq0$ ist.

folgendermaßen verallgemeinert werden. Ist $\lambda=\xi+\eta i$ komplex, so folgt

$$\frac{\alpha^2-\beta^2+2\alpha\beta i}{4\lambda}=\frac{(\alpha^2-\beta^2)\,\xi+2\alpha\beta\eta+[2\alpha\beta\xi-(\alpha^2-\beta^2)\,\eta]\,i}{4(\xi^2+\eta^2)}\,.$$

Nun, solange $\Re\lambda>0$ ist, existiert das Integral $A(x)=\int_{-\infty}^{\infty} e^{-\lambda v^2}e^{\alpha v+\beta vi}\,dv$ in der komplexen λ-Ebene, während in die längs negativer Reelachse gesperrten λ-Ebene der Ausdruck $\sqrt{\frac{\pi}{\lambda}}\exp\frac{\alpha^2-\beta^2+2\alpha\beta i}{4\lambda}$ sich eindeutig regulär verhält. Aber wegen (7) stimmen diese zwei Funktionen längs Wegstückchens $\lambda=\xi>0$ überein. Nach dem Prinzip der analytischen Fortsetzung gilt daher (7) allgemein auch in der Halbebene $\Re\lambda>0$. Also erhält man

$$(8)\qquad |A(\lambda)|=\sqrt{\frac{\pi}{|\lambda|}}\exp\frac{(\alpha^2-\beta^2)\,\xi+2\alpha\beta\eta}{4(\xi^2+\eta^2)}\,,\qquad (\Re\lambda>0)$$

wie auch $\lambda=\xi+\eta i$ $(\xi>0)$ und $\gamma=\alpha+\beta i$ $(\beta\neq 0)$ immer gegeben werden mögen. Wir wollen jetzt diese möglichen einzelnen Fälle umständlich untersuchen.

Fall I. $\alpha=0$, $\beta\gtrless 0$. Es strebt wegen (8)

$$(9)\qquad |A(\lambda)|=\sqrt{\frac{\pi}{|\lambda|}}\exp\left\{\frac{-\beta^2\xi}{4|\lambda|^2}\right\}\to 0\qquad \text{für}\quad \lambda=\xi\to 0\,.$$

Überdies bleibt die Beziehung (9) auch dann noch gültig, wenn die komplexe Variable λ bei der Annäherung an 0 auf den im Innern der rechten Halbebene liegenden Winkelraume beschränkt ist, der von zwei Strahlen $\eta=\pm\xi\cot\delta$ (wo $\delta>0$ beliebig klein ist) gebildet wird (das punktierte Bereich in Fig. I). Ist nämlich $\{\lambda=|\lambda|e^{i\theta}\}$ irgendeine beliebige Punktfolge, die in dem genannten Winkelraum liegt und gegen 0 rückt, so wird

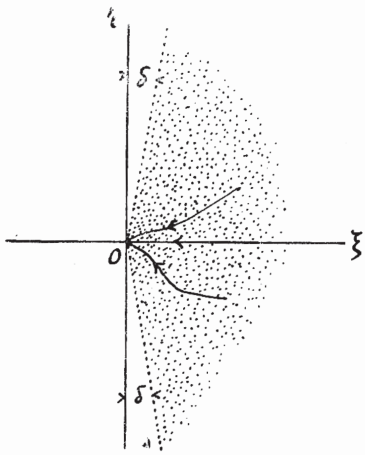

Fig. I $(\alpha=0,\ \beta\gtrless 0)$

$$|A(\lambda)|=\sqrt{\frac{\pi}{|\lambda|}}\exp\left\{\frac{-\beta^2\cos\theta}{4|\lambda|}\right\}.$$

Da aber hierin $\frac{\beta^2}{4}\cos\theta>\frac{\beta^2}{4}\sin\delta=\kappa$ fest und >0 bleibt, so ist

$$|A(\lambda)| < \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{-\frac{\kappa}{|\lambda|}\right\}$$

und folglich es strebt gleichmäßig gegen 0 für $\lambda \to 0$.

Fall II. $|\alpha| = |\beta|$, $\alpha\beta \lessgtr 0$. In diesem Falle aus (8) ergibt sich

$$|A(\lambda)| = \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{\frac{\mp\alpha^2\eta}{2|\lambda|^2}\right\},$$

je nachdem das Produkt $\alpha\beta \lessgtr 0$ ist. Rückt nun $\eta = |\lambda|\sin\theta$ gegen 0 in dem oberen bezw. unteren Winkelraume, der durch je zwei von Strahlen $\eta = \pm\xi\cot\delta$ und $\eta = \pm\xi\tan\delta$ ($\delta > 0$ beliebig klein) gebildet wird (Fig. II), so strebt ebenso auch $A(\lambda) \to 0$. Denn, alsdann gilt

$$\begin{aligned}|A(\lambda)| &= \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{\mp\frac{\alpha^2\sin\theta}{2|\lambda|}\right\} < \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{\frac{-\alpha^2\sin\delta}{2|\lambda|}\right\} \\ &= \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{-\frac{\kappa}{|\lambda|}\right\} \to 0 \qquad \text{für } \lambda \to +0.\end{aligned}$$

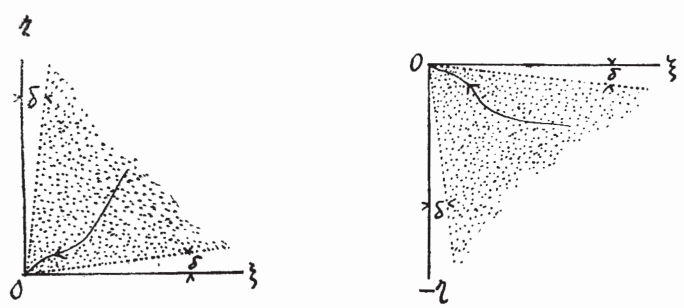

Fig. II ($|\alpha| = |\beta|$, $\alpha\beta \lessgtr 0$)

Fall III. $\beta^2 > \alpha^2$, $\alpha\beta \gtrless 0$. Man bestimme aus $\tan\psi = \dfrac{\beta^2-\alpha^2}{2\alpha\beta}$ den Winkel ψ, derart, daß bei $\alpha\beta \gtrless 0$ bezw. $0 < \pm\psi < \dfrac{\pi}{2}$ gilt. Dreht man die λ $(= \xi + \eta i)$-Ebene um den Winkel $\varphi = \psi \mp \dfrac{\pi}{2}$ mit $\lambda = 0$ als festbleibenden Drehpunkt, so erhält man

$$\begin{aligned}\xi &= \xi'\cos\varphi - \eta'\sin\varphi = \pm\xi'\sin\psi \pm \eta'\cos\psi, \\ \eta &= \xi'\sin\varphi + \eta'\cos\varphi = \mp\xi'\cos\psi \pm \eta'\sin\psi,\end{aligned}$$

wodurch das Zahlenpaar (ξ, η) auf (ξ', η') transformiert wird. Denkt man sich jetzt jeden zu beiden Halbebenen $\xi > 0$, $\xi' > 0$ gemeinsamen Winkelraum, der von je zwei der Strahlen $\eta = \mp\xi\cot\delta$, $\eta' = \pm\xi'\cot\delta$ gebildet wird (Fig. III), so erkennt man wieder, daß

$$|A(\lambda)| < \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{-\left|\frac{\alpha\beta}{2\lambda}\right| \frac{\sin\delta}{\cos\psi}\right\} = \sqrt{\frac{\pi}{|\lambda|}} \exp\left\{-\frac{\kappa}{|\lambda|}\right\} \rightarrow 0$$

für $\lambda \rightarrow 0$ strebt.

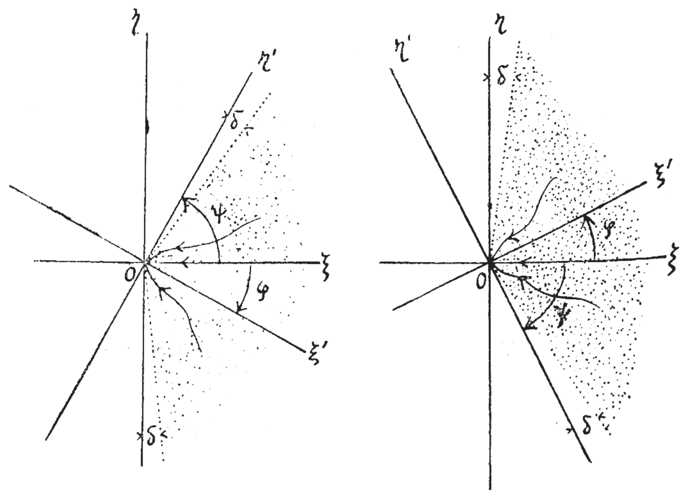

Fig. III ($\beta^2 > \alpha^2$, $\alpha\beta \gtreqless 0$)

Fall IV. $\alpha^2 > \beta^2$, $\alpha\beta \lesseqgtr 0$. Man betrachte wieder das Innere des zu beide Halbebenen $\xi > 0$, $\xi' > 0$ gemeinschaftlich genommenen Teil, wie in Fall III, und den dort liegenden analogen (aber jetzt scharfeckigen) Winkel-

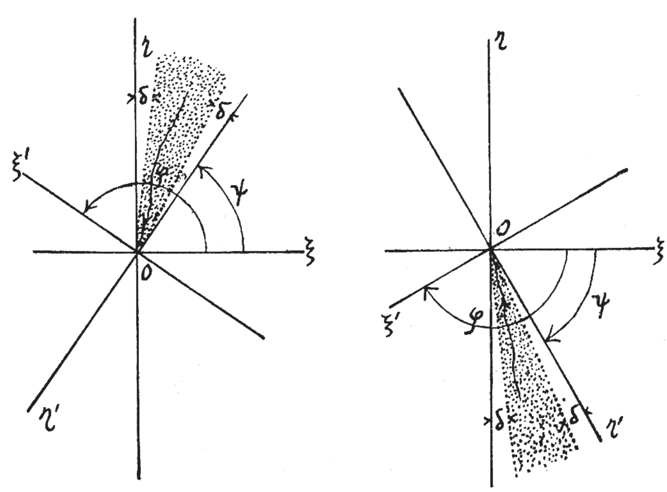

Fig. IV ($\alpha^2 > \beta^2$, $\alpha\beta \lesseqgtr 0$)

raum (Fig. IV). Läßt man nämlich den Drehungswinkel bis zu $\varphi = \psi \pm \frac{\pi}{2}$ mit $\tan\psi = \frac{\beta^2 - \alpha^2}{2\alpha\beta}$ betragen, so gilt nochmals

$$|A(\lambda)| < \sqrt{\frac{\pi}{|\lambda|}}\exp\left\{-\left|\frac{\alpha\beta}{2\lambda}\right|\frac{\sin\delta}{\cos\psi}\right\} = \sqrt{\frac{\pi}{|\lambda|}}\exp\left\{-\frac{\kappa}{|\lambda|}\right\} \to 0$$

für $\lambda \to +0$.

Fügt man alle obige Ergebnisse zusammen, so lehren sie, daß

$$(10) \qquad \begin{cases} A\text{-}v^2\text{-}\displaystyle\int_{-\infty}^{\infty} \exp(\alpha v + \beta v i)\, dv = 0\,, \\ A\text{-}(\log u)^2\text{-}\displaystyle\int_0^{\infty} u^{\alpha-1+\beta i}\, du = 0\,, \end{cases} \qquad (\beta \gtrless 0,\ \alpha \gtreqless 0)$$

gilt, wenn $\lambda = \xi + \eta i$ längs gewißes Weges gegen 0 strebt.

§ 4. Gelegentlich betrachte ich als viertes Beispiel das Integral

$$(11) \qquad \int_\varepsilon^\omega u^{\alpha-1} e^{i\beta u}\, du \qquad \text{für} \quad \varepsilon \to +0\,,\ \omega \to +\infty\,,$$

wobei $\alpha = n + \gamma > 0$, $n = 0, 1, 2, \cdots$ und $0 \leqq \gamma \leqq 1$, $\beta \gtrless 0$ sind.[5] Bekanntlich hat dies schon eine C_k-Summe $(k > \alpha)$, und folglich auch gewöhnliches Abelsches Mittel. In der Tat erlaubt das Abelsche Mittel bei $\alpha > 1$ eine Teilweise-Integration:

$$A(\lambda) = \int_0^\infty e^{-\lambda u} u^{\alpha-1} e^{\beta u i}\, du = \left[\frac{u^{\alpha-1} e^{-\lambda u + \beta u i}}{-\lambda + \beta i}\right]_0^\infty + \frac{\alpha-1}{\lambda - \beta i}\int_0^\infty e^{-\lambda u + \beta u i} u^{\alpha-2}\, du\,,$$

worin der integrierte Teil wegen $\Re\lambda > 0$, $\alpha > 1$ verschwindet.[6] Also ergibt sich nach n maligen wiederholten partiellen Integrationen

$$(12) \qquad A(\lambda) = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{(\lambda-\beta i)^n}\int_0^\infty e^{-\lambda u + \beta u i} u^{\gamma-1}\, du\,,$$

und daraus folgt bei $\gamma = 1$, $\alpha = n+1$ bereits das Resultat, daß

$$(13) \qquad \int_0^\infty e^{-\lambda u} u^n e^{\beta u i}\, du = \frac{n!}{(\lambda-\beta i)^{n+1}} \to \frac{\Gamma(n+1)}{(-\beta i)^{n+1}} \qquad \text{für} \quad \lambda \to 0$$

strebt. Andererseits gilt bei $0 < \gamma < 1$

$$(14) \qquad \int_0^\infty u^{\gamma-1} e^{i\beta u}\, du = \frac{\Gamma(\gamma)}{\beta^\gamma}\exp\frac{\pi}{2}\gamma i\,.$$

Denn, die Funktion $f(z) = z^{\gamma-1}e^{-\beta z}$ $(0 < \gamma < 1,\ \beta > 0)$ ist regulär in demjenigen Bereich S, der durch zwei Stücke $0 < \rho \leqq x < R$ bezw. $0 < \rho \leqq y \leqq R$ auf Reel- bezw. Imaginär-Achse und zwei erste Quadranten der Kreisen

5) Im Falle $\alpha \leqq 0$ wird das Integral bestimmt $+\infty$.

6) Im Falle $\alpha = 1$ gilt schon $A(\lambda) = \dfrac{1}{\lambda - i\beta} \to \dfrac{i}{\beta}$ für $\lambda \to 0$; d. h. $\displaystyle\lim_{\lambda\to+0}\int_0^\infty e^{-\lambda u}\,{\cos \atop \sin}\,\beta u\, du = \begin{cases}0, \\ 1/\beta.\end{cases}$

$|z| = \rho \to 0$, $|z| = R \to \infty$ umgeschlossen wird. Aus das Verschwinden von dem über den Umfang C des Bereiches S erstreckten Cauchyschen Integral folgt ohne weiteres die Formel (14).

Durch Einsetzung in (12) aus (14) erhält man das gesuchte Abelsche Mittel:

$$\lim_{\lambda \to +0} \int_0^\infty e^{-\lambda u} u^{\alpha-1} (\cos \beta u + i \sin \beta u)\, du = \frac{\Gamma(\alpha)}{\beta^\alpha} \exp \frac{\pi}{2} \alpha i, \tag{15}$$

wobei wegen $\alpha = n + \gamma$ der Faktor $\exp \frac{\pi}{2} \alpha i = i^n \exp \frac{\pi}{2} \gamma i$ wird. Z. B. ist für ganzzahliges α

$$A\text{-}\int_0^\infty u^{2p} \begin{matrix}\cos\\ \sin\end{matrix} \beta u\, du = \begin{cases} 0 \\ (-1)^p \underline{|2p}/\beta^{2p+1} \end{cases} \qquad \text{bei} \quad \alpha = 2p+1,$$

oder

$$A\text{-}\int_0^\infty u^{2p-1} \begin{matrix}\cos\\ \sin\end{matrix} \beta u\, du = \begin{cases} (-1)^p \underline{|2p-1}/\beta^{2p} \\ 0 \end{cases} \qquad \text{bei} \quad \alpha = 2p.$$

Ich erwünschte, aber vergebens, das $\boldsymbol{A}\text{-}e^{e^x}$-Mittel des Integrales[7)]

$$\int_0^\infty e^{\alpha x} x^{\beta i}\, dx \equiv \int_0^\infty \exp \{\alpha x + i\beta \log x\}\, dx \qquad (\alpha > 0,\ \beta \gtrless 0) \tag{17}$$

zu finden, was ich noch vielleicht als 0 vermutet habe.

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7) Man kann $\alpha = 1$ annehmen, ohne Beschränkung der Allgemeinheit; Oder wird $e^x = t$ gesetzt, so wird es lediglich $\int_1^\infty (\log t)^{\beta i}\, dt$, eine scheinbar sehr einfache Gestalt.

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ÜBER DIE VERTRÄGLICHKEIT-EIGENSCHAFT DER GEWISSERMASSEN ERWEITERTEN CESÀROSCHEN UND ABELSCHEN LIMITIERUNGSVERFAHREN BEI INTEGRALEN

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In meiner vorstehenden Note „Eine Verallgemeinerung des Abelschen Limitierungsverfahren bei Integralen“ ist nichts in bezug auf seiner Verträglichkeitsbedingung erreicht. Es scheint zwar wertvoll die Bedingungen zu entscheiden, worunter aus dem Vorhandensein eines von beiden Limes, A und $\boldsymbol{A}$, dasjenige des anderes gefolgert werden kann. Da aber dies mir noch nicht gelungen ist[1], soll in der vorliegenden Note nur die Verträglichkeit zwischen gewissermaßen erweiterten Cesàroschen und Abelschen Limitierungsverfahren behandelt werden.

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1) Es sei $\varphi(x, u)$ der Kern der Transformation welche gewöhnliches $\boldsymbol{A}$-Mittel auf unser verallgemeinertes $\boldsymbol{A}$-Mittel führt, also

$$\int_\omega^x \varphi(x, u)\frac{du}{u}\int_a^\infty \exp\left(-\frac{v}{u}\right)s(v)dv \cong \frac{1}{x}\int_a^\infty \exp\left\{-\frac{g(v)}{x}\right\}g'(v)s(v)dv\,,$$

wobei $g(x)$ irgendeine gegebene monoton zunehmende positive Funktion mit $g(\infty)=\infty$ bedeutet, und das Zeichen $\cong$ die für genügend großes x asymptotish bestehende Gleichheit meint. Dann bekommen wir eine Integralgleichung:

$$\int_\omega^x \varphi(x, u)\exp\left(-\frac{v}{u}\right)\frac{du}{u} \cong \frac{1}{x}\exp\left\{-\frac{g(v)}{x}\right\}g'(v)\,,$$

woraus die unbekante Funktion, $\varphi(x, u)$ gefunden werden soll. Aber diese Integralgleichung hat eine ganz verschiedene Gestalt als klassische, und also gehört zu weder Volterraschem noch Fredholmschem Typus. Aber doch die hier eingeführte Integralgleichung ist keineswegs sinnlos. Ist z.B. $g(v)=v$, so lautet $\int_\omega^x \varphi(x, u)e^{-\frac{v}{u}}\frac{du}{u}\cong\frac{1}{x}e^{-\frac{v}{x}}$. Setzt man etwa $\varphi(x, u)=0$ bei $\omega<u<x-1$ und $\varphi(x, u)=\frac{u}{x}$ bei $x-1<u<x$ voraus, so erhält man mit Benutzung des zweiten Mittelwertsatzes für großes x

$$\int_\omega^x = \int_{x-1}^x e^{-\frac{v}{u}}\frac{du}{u} = e^{-\frac{v}{x-1}}\int_{x-1}^\xi \frac{du}{x}+e^{-\frac{v}{x}}\int_\xi^x \frac{du}{x}\cong\frac{1}{x}e^{-\frac{v}{x}}.$$

Also genügt oben vorausgesetztes $\varphi(x, u)$ asymptotisch der Integralgleichung, und sogar sogenannten Toeplitzschen Bedingungen, da sowohl $\int_\omega^x \varphi(x, u)du\cong 1$, als $\varphi(x, u)\cong 0$ für $x\to\infty$ stattfinden. Allgemeiner, multipliziert man beide Seite mit v^n und integriert von $a(=0)$ bis ∞, so erhält man

$$\int_\omega^x u^n\varphi(x, u)du = -\int_0^\infty v^n d\,\exp\left(-\frac{g(v)}{x}\right) = \mu_n\,, \qquad (n=0, 1, 2, \cdots)$$

Hiermit hat man sog. Hausdorff-Hamburgersches Moment-Problem zu auflösen.

§ 1. Man definiere neues Cesàrosches Mittel durch

$$
(1) \qquad \boldsymbol{C}_{\kappa}\text{-}s(x) = \frac{k}{g(x)^k}\int_a^x [g(x)-g(u)]^{k-1} s(u) g'(u)\, du \qquad (k>0)
$$

$$
= \int_a^x \left[1-\frac{g(u)}{g(x)}\right]^k s'(u)\, du\,,
$$
[2]

worin die Funktion $g(x)$ positiv, monoton wachsend, gegen ∞ für $x\to\infty$ strebend, und zweimal differenzierbar ist, dabei allgemein $0 \leq g'(x) < \infty$ bleibt, aber sonst im Falle $g'(a)=\infty$ doch $\lim\limits_{x\to a} xg'(x)<\infty$, also $a=0$, $g'(x)=O\left(\frac{1}{x}\right)$ für $x\to 0$ sein soll. Jedoch darf $g(x)$ sogenannt stückweise glatt sein, oder sogar Sprünge machen[3]. Wenn es abzählbar unendlich vielen Sprungsstellen geben, so braucht man das Integral nur im Lebesgueschen Sinne zu verstehen. Für $g(x)=x$ stimmt unser $\boldsymbol{C}_k\text{-}s(x)$ mit übriches Cesàrosches Mittel $C_k\text{-}s(x)$ überein. Nun soll die Verträglichkeit

$$
(2) \qquad \left(\lim_{x\to\infty} C_k\text{-}s(x) = l\right) \rightleftarrows \left(\lim_{x\to\infty} \boldsymbol{C}_k\text{-}s(x) = l\right)
$$

untersucht werden. Da es bei gebrochenes k etwas verwickelt ist, so beschäftige ich mich nur mit dem Falle $k=1$, d.h. dem arithmetischen Mittel.

Es sei $\varphi(x, u)$, der Kern der Transformation, wodurch $C_1\text{-}s(x)$ auf $\boldsymbol{C}_1\text{-}s(x)$ transformiert werde:

$$
\int_a^x \varphi(x,u)\frac{du}{u}\int_a^u s(v)\,dv = \int_a^x s(v)\,dv \int_v^x \varphi(x,u)\frac{du}{u}
$$

$$
\simeq \frac{1}{g(x)}\int_a^x s(v)g'(v)\,dv \qquad \text{bei} \quad x\to\infty .
$$

Daraus läßt sich zeigen, daß

$$
(3) \qquad \int_v^x \varphi(x,u)\frac{du}{u} \simeq \frac{g'(v)}{g(x)} .
$$

Differenziert man partiell nach v, so kommt

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[2] Wenn man das Integral als Stieltjesch verstehe, so tritte die unendliche Reihe $s(n)=\sum_{\nu=0}^{\nu=n} a_\nu$. auf. Oder dürfe für das Integral $s(x)=\int_a^x f(u)du$ auch $g(u)$ als Stufenfunktion angenommen werden; Jedoch müsse alsdann das Argument unten etwas umgeschriebt werden.

[3] Z. B. sei $g(x)=\sqrt{x}$ für $0<x<1$, sei $g(x)=5$ für $1<x<10$ und sei $g(x)=x$ für $10<x<\infty$. Wenn der $C_1\text{-}s(x)$-Limes existiert, welcher, ohne Allgemeinheit zu verloren, als 0 vorausgesetzt werden kann, so ist bekanntlich $s'(x)=o(x)$. Daher gilt für genügend großes x

$$
\boldsymbol{C}_1\text{-}g(x)\text{-}s(x) = \int_0^1\left(1-\frac{\sqrt{u}}{x}\right)s'(u)\,du + \int_1^{10}\left(1-\frac{5}{x}\right)s'(u)\,du + \int_{10}^x \left(1-\frac{u}{x}\right)s'(u)\,du
$$

$$
\cong s(x) + O\left(\frac{1}{x}\right) - \frac{1}{x}\int_{10}^x us'(u)\,du \cong C_1\text{-}s(x) .
$$

$$(4)\qquad -\frac{\varphi(x,v)}{v} \simeq \frac{g''(v)}{g(x)}, \quad \text{d.h.} \quad \varphi(x,v) \simeq -\frac{vg''(v)}{g(x)}.$$

Wird aber dies in (3) eingesetzt, so ergibt sich

$$(5)\qquad -\int_v^x \frac{g''(v)}{g(x)}dv = \frac{g'(v)}{g(x)} - \frac{g'(x)}{g(x)}.$$

Sind (3) und (5) asymptotisch gleichwertig, so muß

$$(6)\qquad \frac{g'(x)}{g(x)} = o(1) \qquad \text{für} \quad x \to \infty$$

bestehen. Ersetzt man nun anstatt (4)

$$(7)\qquad \psi(x,v) = -\frac{vg''(v)}{g(x)} + \frac{g'(x)}{g(x)},$$

so strebt wegen (6) $\psi(x,v) \to 0$ für $x \to \infty$ bei festes v. Ferner gilt

$$\begin{aligned}\int_a^x \psi(x,v)\,dv &= (x-a)\frac{g'(x)}{g(x)} - \frac{1}{g(x)}\int_a^x vg''(v)\,dv \\ &= \frac{1}{g(x)}\Big\{(x-a)g'(x) - xg'(x) + ag'(a) + g(x) - g(a)\Big\} \to 1\end{aligned}$$

für $x \to \infty$, infolge der zu $g(x)$ gemachten Voraussetzung und (6) wegen.

Man kann sogar erweisen, daß

$$\int_a^x |\psi(x,v)|\,dv < K$$

entsteht, wenn schließlich (etwa für $x > x_0$) $g''(v) \leq 0$ ist, oder sonst schließlich $g''(v) \geq 0$ und $xg'(x)/g(x)$ beschränkt bleibt. Deshalb bestehen alle sämtlich unter folgenden Bedingungen:

(8) $\overline{\lim}\, xg'(x)/g(x) < \infty$,

(9) die Anzahl des Zeichenwechsel von $g''(x) =$ endlich[4].

Unter Bedingungen (8), (9) liefert also $\psi(x,u)$ gewiß eine Toeplitzsche (konvergenzerhaltende) Transformation. Ist daher $s(x)$ etwa C_1-limitierbar zum Wert l, so stimmt auch

$$(10)\qquad \int_a^x \psi(x,u) C_1\text{-}s(u)\,du \to l \qquad \text{für} \quad x \to \infty.$$

Andererseits ist

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[4] Ich ziehe in Zweifel, daz zwischen (8) und (9) es etwaige Beziehung geben und folglich von einander nicht unabhängig sein mögen. Z. B. falls $\lim_{x\to\infty} xg'(x)/g(x) = \mu \neq 1$ vorhanden ist, und überdies die Umkehrung des l' Hospitalsch Regel zulässig ist (was aber manchmal fehlt, vgl. dazu § 2 Beispiel 18), so hat man $\lim\,[g(x)+xg''(x)]/g(x) = \mu$, woraus $xg''(x)/g(x) \to \mu-1 \neq 0$ folgt und deswegen $g''(x)$ sein Zeichen schließlich bewahren soll.

$$(11)\quad \int_a^x \psi(x,u)\frac{du}{u}\int_a^u s(v)\,dv = \int_a^x s(v)\,dv \int_v^x \psi(x,u)\frac{du}{u}$$
$$= \frac{1}{g(x)}\int_a^x s(v)g'(v)\,dv - \frac{xg'(x)}{g(x)}\left[\frac{1}{x}\int_a^x s(v)\,dv - \frac{1}{x}\int_a^x s(v)\log\frac{x}{v}\,dv\right].$$

Das erste Integral in der rechten Seite ist nichts anderes als gewünschte $\boldsymbol{C}_1$-$g(x)$-$s(x)$, während zwei Integrale in der eckigen Krammer eben C_1-$s(x)$ bezw. Höldersche Mittel H_2-$s(x)$ sind, somit einander gleich werden, falls der C_1-Limes existiert. Also folgt aus Vorhandensein des C_1-Limes unter Bedingung (8), daß Ausdruck (11) gegen den $\boldsymbol{C}_1$-$g(x)$-Limes für $x \to \infty$ strebt. Dies mit (10) zusammengesetzt, kann man schließen, daß der obere Übergang $\to$ in (2) alsdann zulässig ist, und formulieren den

Satz 1. Die Funktion $g(x)$ sei im ganzen Integrationsintervalle $a \leqq x < \infty$ erklärt, und dort positiv, monoton gegen ∞ wachsend, und ferner zweimal differenzierbar, dabei $\lim\limits_{x\to\infty} g'(x)/g(x) = 0$ und durchaus $g''(x) \leqq 0$ gelte, oder sonst $xg'(x)/g(x)$ beschränkt bleibe, und im allgemeinen $g''(x)$ schließlich sein Zeichen bewahre. Dann kann man stets das Vorhandensein des $\boldsymbol{C}_1$-$g(x)$-$s(x)$ behaupten, sobald C_1-$s(x)$ existiert.

§ 2. Beispielsweise schreibe ich folgende Tabelle manches $g(x)$. Man kann leicht sich überzeugen, daß für jedes von (1) bis (9) Satz 1 gilt.

Nr.	Anfangs-Punkt a	$g(x)$	$g'(x)$	$\lim xg'(x)/g(x)$
1	0	$x^c (c>0)$	cx^{c-1}	c
2	1	$\log x$	$1/x$	0
3	e	$\log\log x$	$1/x \log x$	0
4	1	$x \log x$	$1+\log x$	1
5	0	$x[1-\exp(-x^2)]$	$1+(2x^2-1)\exp(-x^2)$	1
6	0	$x^c \exp(-1/x)$	$g(x)(1+cx)/x^2$	c
7	0	$x^c \exp(1+x)^{-1}$	$g(x)[cx^{-1}-(x+1)^{-2}]$	c
8	0	$\log\cosh x$	$\tanh x$	1
9	0	$x \tan^{-1} x$	$\tan^{-1}x + x/(1+x^2)$	1
10	$-\infty$	e^x	e^x	∞
11	$-\infty$	$\exp(\exp x)$	$e^x \exp(\exp x)$	∞
12	1	$x^{\log x}$	$2x^{\log x}\log x/x$	∞
13	1	$(\log x)^x$	$g(x)[\log\log x + 1/\log x]$	∞
14	1	$(\log x)^{\log x}$	$g(x)[\log\log x+1]/x$	∞
15	0	$e^x[5-(\cos 2x+2\sin 2x)]$	$10e^x \sin^2 x$	$\underline{\lim}=0,\ \overline{\lim}=\infty$

In Absicht der Erläuterung füge ich noch einige Beispiele hinzu.

16. Es sei

$$g(x) = 2\sqrt{x} + \int_0^x \frac{\cos u}{\sqrt{u}}\,du + \int_0^x (x-u)\frac{\sin u}{\sqrt{u}}\,du \qquad (a=0).$$

Hier ist $g(x)\to\infty$ für $x\to\infty$, da bekanntlich

$$\begin{matrix} c(x) \\ s(x) \end{matrix} = \int_0^x \begin{matrix}\cos \\ \sin\end{matrix}\, u \frac{du}{\sqrt{u}} \to \sqrt{\frac{\pi}{2}} \qquad \text{für} \quad x\to\infty$$

und demnach

$$\int_0^x (x-u)s'(u)\,du = xC_1\text{-}s(x) \simeq \sqrt{\frac{\pi}{2}}\,x$$

strebt. Ferner gilt

$$g'(x)=\frac{1+\cos x}{\sqrt{x}}+\int_0^x \frac{\sin u}{\sqrt{u}}\,du > 0\,,$$

und
$$g''(x) = -\frac{1+\cos x}{2\sqrt{x^3}} \leqq 0\,,$$

so daß $g(x)$ gewiß monoton zunimmt. Obgleich $g'(x)\to\infty$ für $x\to 0$ ist, jedoch ist $\lim\limits_{x\to 0} xg'(x)=0$. Weiter

$$\frac{xg'(x)}{g(x)} \simeq \frac{\sqrt{x}(1+\cos x)+x\sqrt{\pi/2}}{2\sqrt{x}+(1+x)\sqrt{\pi/2}} \to 1 \qquad \text{für} \quad x\to\infty\,.$$

Für dieses $g(x)$ ist also Satz 1 gültig.

17. Folgendes Beispiel zeigt, daß das Bestehen von Bedingung (9) ausnahmsweise nicht notwendig ist. Es sei $g''(x)=\dfrac{\sin x}{x}\gtrless 0$, so daß $g'(x)=\displaystyle\int_0^x \frac{\sin v}{v}\,dv \to \frac{\pi}{2}$, $g(x)=\displaystyle\int_0^x g'(v)\,dv = xC_1\text{-}g'(x) \simeq \frac{\pi}{2}x$, $\dfrac{xg'(x)}{g(x)}\simeq 1$ für $x\to\infty$ streben, und deswegen etwa für $x>x_0$, $g'(x)<\pi, g(x)>\left(\frac{\pi}{2}-1\right)x$ gehalten werden. Daraus folgt für $\omega > x_0$

$$\int_\omega^x |\psi(x,v)|\,dv = \frac{1}{g(x)}\int_\omega^x |g'(x)-vg''(v)|\,dv < \frac{1}{x}\int_\omega^x \frac{\pi+1}{\frac{\pi}{2}-1}\,dv < K\,,$$

während natürlich $\displaystyle\int_0^\omega |\psi(x,v)|\,dv < K'$ ist. Also bleibt $\displaystyle\int_0^x |\psi(x,v)|\,dv$ noch beschränkt, damit dieses $\psi(x,v)$ sicher eine Toeplitzsche Transformation herstellt. Mit solches $g(x)$ steht Satz 1 doch fest.

18. Bei folgendes Beispiel nebst immer wechselndem sg $g''(x)$ wird Satz 1 allerdings unannehmber. Gegeben sei $g(x)=x+\sin x$, so daß $g'(x)=1+\cos x \geqq 0$, $g''(x)=-\sin x \gtrless 0$. Dann gilt $\dfrac{xg'(x)}{g(x)}=\dfrac{x(1+\cos x)}{x+\sin x}\simeq 1 + \cos x$, so ist $\underline{\lim}=0$, $\overline{\lim}=2$ und $xg'(x)/g(x)$ beschränkt. Aber mit dieses $g(x)$ gestattet $\psi(x,v)$ keine Toeplitzsche Transformation. Es ist zwar

$$\int_0^x |\psi(x,v)|\,dv = \int_0^\omega + \int_\omega^x = \text{(i)}+\text{(ii)}\,,$$

wobei mit genügend großes festes $\omega > x_0$ auch (i) beschränkt ist, aber doch nicht so für

$$\text{(ii)} \simeq \frac{1}{x}\int_{\omega}^{x} |1+\cos x + v \sin v|\, dv\,.$$

Denn, setzt man $\left[\frac{x}{2\pi}\right] = m$, $\left[\frac{\omega}{2\pi}\right] = n$, $\left[\frac{v}{2\pi}\right] = \nu$ und $v = 2\pi\nu + t$, so erhält man

$$\text{(ii)} \simeq \frac{1}{x}\sum_{\nu=n}^{m}\int_{0}^{2\pi} |1+\cos x + (2\nu\pi + t)\sin t|\, dt$$

$$> \frac{1}{x}\sum_{\nu=n}^{m}\left(\int_{0}^{\pi}\Big\{1+\cos x + (2\nu\pi+t)\sin t\Big\}dt + \int_{\eta_\nu}^{\pi-\eta_\nu}\Big\{-1-\cos x + (2\nu\pi+t)\sin t\Big\}dt\right),$$

wobei $\sin \eta_\nu = \dfrac{1+\cos x}{2\nu\pi} < \varepsilon$ (beliebig klein) für $\omega > x_0$ (genügend groß) ist. Es sei $\eta = \underset{n<\nu<m}{\operatorname{Max}}\, \eta_\nu\, (< 1/\nu\pi)$. Alsdann besteht

$$\text{(ii)} > \frac{2}{x}\sum_{\nu=n}^{m}\int_{\eta}^{\pi-\eta}(2\nu\pi+t)\sin t\, dt \simeq \frac{2}{x}\sum_{\nu=n}^{m} 2\nu\pi\Big[-\cos t\Big]_{\eta}^{\pi-\eta}$$

$$\simeq \frac{4\pi}{x}\Big((m^2-n^2) \simeq \frac{x}{\pi}\Big(1-\frac{\omega^2}{x^2}\Big) \to \infty \qquad \text{für} \quad x\to\infty\,;$$

also ist $\int_0^x |\psi(x, v)|\, dv$ gewiß unbeschränkt. Daher kann nicht $\psi(x, v)$ eine Toeplitzsche, und Satz 1 nunmehr zulässig sein.

19. Auf Probe nehme ich die Stufenfunktion $[x]$ als $g(x)$ an. Es sei $[x] = n$ und $[u] = m$. Man erhält nach Definition

$$\boldsymbol{C}_1\text{-}[x]\text{-}s(x) = \int_0^x\Big(1-\frac{m}{n}\Big)s'(u)\,du = \sum_{m=0}^{n-1}\Big(1-\frac{m}{n}\Big)\int_m^{m+1} s'(u)\,du$$

$$= \sum_{m=0}^{n-1}\Big(1-\frac{m}{n}\Big)\Big\{s(m+1)-s(m)\Big\} = \frac{1}{n}\sum_{m=1}^{n} s(m)\,.$$

Konvergiert nun dies für $x\to\infty$ nach demjenigen das C_1-$s(x)$ strebenden Wert, dessen Existenz vorausgesetzt ist? Ohne allgemeine Betrachtung erprobe ich es für den Fall $a = 0$, $s'(x) = e^{ix}$, $s(x) = i(1-e^{ix})$, dessen C_1-Limes ersichtlich i ist. Nun lautet

$$\boldsymbol{C}_1\text{-}[x]\text{-}s(x) = \frac{i}{n}\sum_{m=1}^{n}(1-e^{im}) = i - \frac{i}{n}\Big(\frac{e^i - e^{i(n+1)}}{1-e^i}\Big) \to i$$

für $n = [x] \to \infty$. Also ist Satz 1 noch bejahnend.

§ 3. Umgekehrt wollen wir den Übergang $\leftarrow$ von (2) beurteilen. Es sei aber jetzt vorausgesetzt, daß die Funktion $y = g(x)$ zwar eigentlich monoton wachsend ist, d.h. ohne beständig $g'(x) = 0$ zu sein, aber übrige Umstände bleibend wie in § 1 besagt. Alsdann ist ihre Umkehrfunktion $x = g^{-1}(y) = h(y)$ mit $h(b) = a$, $h(\infty) = \infty$ versehen, und wieder eigentlich monoton wachsend. Werden ferner $v = g(u)$ und $s(u) = s(h(v)) = S(v)$ gesetzt, so kommen

$$\begin{aligned}\mathbf{C}_k\text{-}s(x) &= \frac{k}{g(x)}\int_a^x \left(1-\frac{g(u)}{g(x)}\right)^{k-1} s(u)g'(u)\,du \\ &= \frac{k}{y}\int_b^y \left(1-\frac{v}{y}\right)^{k-1} S(v)\,dv \equiv C_k\text{-}S(y)\,. \\ C_k\text{-}s(x) &= \frac{k}{x}\int_a^x \left(1-\frac{u}{x}\right)^{k-1} s(u)\,du \\ &= \frac{k}{h(y)}\int_b^y \left(1-\frac{h(v)}{h(y)}\right)^{k-1} S(v)h'(v)\,dv \equiv \mathbf{C}_k\text{-}S(y)\,;\end{aligned}$$

ins besondere

$$\mathbf{C}_1\text{-}s(x) = \frac{1}{g(x)}\int_a^x s(u)g'(u)\,du = \frac{1}{y}\int_b^y S(v)\,dv = C_1\text{-}S(y)\,,$$

$$C_1\text{-}s(x) = \frac{1}{x}\int_a^x s(u)\,du = \frac{1}{h(y)}\int_b^y S(v)h'(v)\,dv = \mathbf{C}_1\text{-}S(y)\,.$$

Man braucht den Übergang zwischen folgenden Aussagen A und B:

$$(A)\quad C_1\text{-}s(x) = \mathbf{C}_1\text{-}S(y) \to l\,, \qquad (B)\quad \mathbf{C}_1\text{-}s(x) = C_1\text{-}S(y) \to l$$

zu untersuchen. Nach Satz 1 kann man direkt schließen, daß

(I) Wenn $\overline{\lim}\, xg'(x)/g(x)$ endlich, und sg $g''(x)$ schließlich (etwa für $x > x_0$) ungeändert bleibt, so ist $A \to B$ wahr;

(II) Wenn $\overline{\lim}\, yh'(y)/h(y)$ endlich, und sg $h''(y)$ schließlich ungeändert bleibt, so ist $B \to A$ wahr.

Da aber offenbar $\dfrac{y}{h(y)}h'(y) = \dfrac{g(x)}{x}\dfrac{1}{g'(x)}$ ist, so gelten

$$\overline{\lim}\,\frac{xg'(x)}{g(x)} = \left[\underline{\lim}\,\frac{yh'(y)}{h(y)}\right]^{-1} = M \leqq \infty\,,$$

und
$$\underline{\lim}\,\frac{xg'(x)}{g(x)} = \left[\overline{\lim}\,\frac{yh'(y)}{h(y)}\right]^{-1} = m \geqq 0\,,$$

Überdies haben $g''(x)$ und $h''(y)$ stets verschiedenes Zeichen, weil

$$\frac{h''(y)}{g''(x)} = -\frac{h'(y)}{g'(x)^2} = -\frac{h'(y)^2}{g'(x)}$$

sind. Deshalb falls eines von ihnen sein Zeichen bewahrt, so ist das anderes ebenso beschaffen.

Aus diese Erwägungen kommt der folgende Satz hervor:

Satz 2. Die Bewahrenheit des Zeichens sei vorausgesetzt. Alsdann hat man folgende Alternierende:

(i) Im Falle $m > 0$, $M < \infty$ sind beide Übergänge $A \rightleftarrows B$ wahr; d.h. C_1- und $\mathbf{C}_1$-Verfahren sind mit einander äquivalent.

(ii) Im Falle $m = 0$, $M < \infty$ kann nur der Übergang $A \to B$ feststellt werden: das $\mathbf{C}_1$-Verfahren ist umfassender als gewöhnliches C_1-Verfahren.

(iii) Im Falle $m>0$, $M=\infty$ ist nur der Übergang $B\to A$ wahrhaft, also wirkt solches $\boldsymbol{C}_1$-Verfahren schwächer als C_1-Verfahren.

(iv) Im Falle $m=0$, $M=\infty$ kann im allgemeinen nichts auf Übergänge $A\rightleftarrows B$ besagt werden.

Ammerkung. Ist z.B. $a=0$, $s(u)=(1-e^{iu})\,i$, so hat $\boldsymbol{C}_1$-$(x+1)\log(x+1)$-$s(x)$ wie C_1-$s(x)$ denselben Limes i, während $\boldsymbol{C}_1$-(e^x-1)-$s(x)$ keinen hat. Da aber für $a=0$, $s(u)=\exp ie^u$ zwar $\boldsymbol{C}_1$-(e^x-1)-$s(x)\to 0$ bei $x\to\infty$ strebt, soll es auch C_1-$s(x)\to 0$ sein. Nach Satz 2 erweitert kein $\boldsymbol{C}_1$-$g(x)$-Verfahren in Fallen (i) und (iii) das Geltungsgebiet über dasjeniges des C_1-Verfahren hinaus. Das $\boldsymbol{C}_1$-$g(x)$-Verfahren im Falle (ii) allein dient den Konvergenzbereich zu erweitern.

§ **4.** Zuletzt beweisen wir den

Satz 3. Es sei $g(x)$ irgendeine positive monoton gegen ∞ zunehmende Funktion[5]. Dann kann es aus der Voraussetzung

$$\boldsymbol{C}_\kappa\text{-}g(x)\text{-}s(x)=\frac{k}{g(x)^k}\int_a^x[g(x)-g(u)]^{k-1}g'(u)s(u)\,du\to l\qquad(x\to\infty)$$

immer zum Schluß

$$\boldsymbol{A}\text{-}g(x)\text{-}s(x)=\lambda\int_a^\infty\exp\{-\lambda g(u)\}\,g'(u)s(u)\,du\to l\qquad(\lambda\to+0)$$

wohl gelingen.

Beweis. Wird das Parameter $\lambda=1/g(x)$ genommen, damit zu $\lambda\to+0$ nun $x\to\infty$ entspricht, und

$$\exp\left\{-\frac{g(u)}{g(x)}\right\}\cdot\frac{g'(u)}{g(x)}=\Phi(x,u)$$

gesetzt, so lautet $$\boldsymbol{A}\text{-}g(x)\text{-}s(x)=\int_a^\infty\Phi(x,u)s(u)\,du\,.$$

Offenbar ist $\Phi(x,u)$ Toeplitzsch. Setzt man ferner

$$\Psi(x,u)=\frac{1}{\Gamma(k+1)}\Phi(x,u)\left\{\frac{g(u)}{g(x)}\right\}^k,$$

so kommt

$$\begin{aligned}&\int_a^\infty\Psi(x,u)\boldsymbol{C}_k\text{-}s(u)\,du\\&=\frac{1}{\Gamma(k+1)}\int_a^\infty\exp\left\{-\frac{g(u)}{g(x)}\right\}\left\{\frac{g(u)}{g(x)}\right\}^k\frac{g'(u)}{g(x)}du\int_a^u\frac{k\{g(u)-g(v)\}^{k-1}}{g(u)^k}g'(v)s(v)\,dv\\&=\frac{1}{g(x)}\int_a^\infty g'(v)s(v)\,dv\int_v^\infty\exp\left\{-\frac{g(u)}{g(x)}\right\}\left\{\frac{g(u)-g(v)}{g(x)}\right\}^{k-1}\frac{g'(u)}{g(x)}\frac{du}{\Gamma(k)}.\end{aligned}$$

---

5) Jetziges $g(x)$ ist ganz frei von der in vorigen Sätzen angelegten anderen Bedingungen, damit e. g. $xg'(x)/g(x)$ unbeschränkt, oder sg $g''(x)$ beliebig wechselnd sein darf.

Das innere Integral kann durch die Substitution $\{g(u)-g(v)\}/g(x)=t$ sofort in der Form

$$\frac{1}{\Gamma(k)}\exp\left\{-\frac{g(v)}{g(x)}\right\}\int_0^\infty e^{-t}t^{k-1}dt=\exp\left\{-\frac{g(v)}{g(x)}\right\}$$

gebracht werden. Folglich gilt

$$\int_a^\infty \Psi(x,\,u)\boldsymbol{C}_k\text{-}s(u)du=\int_a^\infty \exp\left\{-\frac{g(v)}{g(x)}\right\}\frac{g'(v)}{g(x)}s(v)dv=\boldsymbol{A}\text{-}g(x)\text{-}s(x)\,.$$

Ins besondere bei $s(u)=1$ ergibt sich für $x\to\infty$

$$\int_a^\infty \Psi(x,\,u)du=\int_a^\infty |\Psi(x,\,u)|\,du\to 1\,,$$

und $\lim\limits_{x\to\infty}\Psi(x,\,u)=0$ gleichmäßig bei festes u, woraus folgt, daß $\Psi(x\;u)$ wieder eine Toeplitzsche Transformation darstellt. Existiert deswegen $\boldsymbol{C}_k\text{-}g(x)\text{-}s(x)\to l$ für $x\to\infty$, so strebt ebenfalls $\boldsymbol{A}\text{-}g(x)\text{-}s(x)\to l$ für $x\to\infty$ $(\lambda\to+0)$, w. z. b. w.

Zusatz. Es sei gewöhnliches $C_k\text{-}s(x)$ für $k\leq 1$ vorhanden und $=l$. Alsdann folgt bekanntlich $C_1\text{-}s(x)\to l$. Befridigt $g(x)$ die Bedingungen (8) und (9), so folgt nach Satz 1, daß $\boldsymbol{C}_1\text{-}s(x)\to l$ und woraus wegen Satzes 3, daß $\boldsymbol{A}\text{-}g(x)\text{-}s(x)\to l$.

Zum Schluß sei noch ein Beispiel beigefügt. Ersichtlich ist das Integral

$$s(x)=\int_1^x\frac{du}{u^{1-i}}=i(1-x^i)\quad(\text{wo }\arg u=0\text{ sei})$$

divergent für $x\to\infty$, und folglich weder C_1-limitierbar noch A-limitierbar, da $s'(x)=O\left(\frac{1}{x}\right)$ gilt. Dessengeachtet läßt sein $\boldsymbol{C}_1$-$\log x$-Limes sich festsetzen wie

$$\begin{aligned}\frac{1}{\log x}\int_1^x s(u)\frac{du}{u}&=\frac{i}{\log x}\int_1^x\left(\frac{1}{u}-\frac{1}{u^{1-i}}\right)du\\&=i-\frac{x^i-1}{\log x}\to i\qquad\text{für}\quad x\to\infty\,.\end{aligned}$$

Also genügt unser $\boldsymbol{C}_1$-$\log x$-Verfahren irgendwie der Erweiterungsbedingung. Sogar ist obiges $s(x)$ nach Satz 3 schon $\boldsymbol{A}$-$\log x$-limitierbar zu demselben Wert i (vgl. dazu meine vorstehende Note § 1). Daraus aber nicht folgt das Vorhandensein des gewöhnlichen C_1-Limes. Denn, für jetziges $g(x)$ ist zwar $\underline{\lim}\, xg'(x)/g(x)=0$, und also als zweite Fall von Satz 2 ist der Übergang $\boldsymbol{C}_1\to C_1$ allgemein unzulässig.

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ON THE PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH CONSTANT COEFFICIENTS

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We consider the linear homogeneous partial differential equation of second order with constant coefficients[1)]

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+H\frac{\partial u}{\partial x}+K\frac{\partial u}{\partial y}+Lu=0. \qquad (1)$$

In particular, when it is of parabolic type, i. e. if $B^2-4AC=0$, then by the familiar transformation $\xi=y-\alpha x$, $\eta=y$, α being the equal root of the equation $A\gamma^2-B\gamma+C=0$, we have

$$\frac{\partial^2 u}{\partial \eta^2}+2a\frac{\partial u}{\partial \xi}+2b\frac{\partial u}{\partial \eta}+cu=0, \qquad (2)$$

where $a=\dfrac{2AK-BH}{B^2}$, $b=\dfrac{K}{2C}$, $c=\dfrac{L}{C}$.[2)] Or, on writing

$$b^2-c=-\mu^2(\neq 0),^{3)} \quad \eta=\frac{t}{\mu},\quad \frac{a}{\mu^2}=h,\quad \frac{b}{\mu}=k \quad \text{and} \quad u=ve^{-kt}, \qquad (3)$$

we obtain

$$\frac{\partial^2 v}{\partial t^2}+2h\frac{\partial v}{\partial \xi}+v=0. \qquad (4)$$

Let supplementary conditions be such that, in regard to (2)

$$u=f(\xi),\quad \frac{\partial u}{\partial \eta}=F(\xi), \qquad \text{when} \quad \eta=0, \qquad (5)$$

which become for (4)

$$v=f(\xi),\quad \frac{\partial v}{\partial t}=\frac{F(\xi)+bf(\xi)}{\mu}\equiv g(\xi), \qquad \text{when} \quad t=0, \qquad (6)$$

where it is assumed that $f(\xi)$ as well as $F(\xi)$ (and so also $g(\xi)$) all permit

1) The constants as well as the variables are supposed to be complex in general.

2) Here we have assumed as $BC\neq 0$; but if one of B and C (or A) vanishes, so also the other must vanish because of $B^2=4AC$, and then the original equation (1) already has the form (2) or alike.

3) The case $\mu=0$ i. e. $b^2=c$ shall be considered later on.

Taylor's expansion in the vicinity of $\xi=0$.[4)] If the solution of (4) satisfying (6) could be found, it should contain two arbitrarily chosen functions $f(\xi)$ and $g(\xi)$, so that it might be regarded as the general solution of (1), if the letters be put back.

For our purpose, we think the equation of hyperbolic type

$$\frac{\partial^2 v}{\partial t^2}-\varepsilon^2\frac{\partial^2 v}{\partial \xi^2}+2h\frac{\partial v}{\partial \xi}+v=0,$$

whose solution is obtainable by Riemann's method with Bessel function. Now making $\varepsilon\to 0$, we get, as the required solution, superficially

$$v=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)V_n(t)-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)\frac{V_{n+1}(t)}{t}, \tag{7}$$

where $V_n(t)$ stands for

$$V_n(t)=\left[\frac{d^n}{d\zeta^n}\cos(\sqrt{1+2\zeta}\,t)\right]_{\zeta=0}, \tag{8}$$

and consequently

$$V_0(t)=\cos t,\ \ V_1(t)=-t\sin t,\ \ V_2(t)=-t_2\cos t+t\sin t,$$
$$V_3(t)=3t^2\cos t+(t^3-3t)\sin t,\ \ V_4(t)=(t^4-15t^2)\cos t-(6t^3-15t)\sin t,$$
$$V_5(t)=-(10t^4-105t^2)\cos t-(t^5-45t^3+1054t)\sin t,\ \text{and so on.}$$

The arguement t being complex in general, if $t=\theta+\tau\sqrt{-1}$ (θ, τ real), so it must be understood that

$$\cos t=\cos\theta\cdot\cosh\tau-i\sin\theta\sinh\tau,\ \ \sin t=\sin\theta\cdot\cosh\tau+i\cos\theta\cdot\sinh\tau.$$

Or expanding $V_n(t)$ in power series, we get

$$V_n(t)=\sum_{l=n}^{\infty}\frac{(-1)^l t^{2l}}{(2l)!}\left[\frac{d^n}{d\zeta^n}(1+2\zeta)^l\right]_{\zeta=0}=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}\underline{|m+n}}{\underline{|m}\ \underline{|2m+2n}}2^n t^{2m+2n}$$
$$=\frac{(-1)^n 2^n\underline{|n}}{\underline{|2n}}t^{2n}\left\{1-\frac{t^2}{(2n+1)\cdot 2\cdot\underline{|1}}+\frac{t^4}{(2n+1)(2n+3)\cdot 2^2\cdot\underline{|2}}-\frac{t^6}{(2n+1)(2n+3)(2n+5)\cdot 2^3\cdot\underline{|3}}+\cdots\right\}, \tag{9}$$

which is evidently convergent in the whole t-plane, and thus defines an integral transcendental function. Moreover $V_n(t)$ satisfies the following differential equation[5)]

4) If $f(\xi)$ or $F(\xi)$ behaves as $1/\xi$ at $\xi=0$, so that not regular, we may put $\xi=\xi'+\beta$; now that $\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial \xi'}$, and $f(\xi')$, $F(\xi')$ become regular at $\xi'=0$, the same treatment is still possible.

5) The equation (10) is different from ordinary equations, such as Legendre, Bessel, Gauss; and therefore it seems to deserve consideration in detail.

$$\frac{d^2y}{dt^2}-\frac{2n}{t}\frac{dy}{dt}+\left(1+\frac{2n}{t^2}\right)y=0 \tag{10}$$

and besides the following identities

$$\frac{V_n}{t^2}-\frac{V_n'}{t}=V_{n-1} \tag{11}$$

and

$$V_n''+V_n=-2nV_{n-1}. \tag{12}$$

Using these relations, we can show that the expression (7) really satisfies the equation (4). In fact, upon differentiating (7) partially with respect to t twice, we have

$$\frac{\partial^2 v}{\partial t^2}=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)V_n''(t)-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)\left[\frac{V_{n+1}''(t)}{t}-2\frac{V_{n+1}'(t)}{t^2}+2\frac{V_{n+1}(t)}{t^3}\right].$$

Adding this to (7), we get

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2}+v=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)[V_n''(t)+V_n(t)]-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)\\ \times\left[\frac{V_{n+1}''(t)}{t}-2\frac{V_{n+1}'(t)}{t^2}+2\frac{V_{n+1}(t)}{t^3}+\frac{V_{n+1}(t)}{t}\right],\end{aligned}$$

in which on account of (12) and (11) the first square bracket reduces to $-2nV_{n-1}(t)$, while the second to $-\frac{2n}{t}V_n(t)$. Or, on factorizing $-2h$, it becomes

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2}+v&=-2h\left[\sum_{m=0}^{\infty}\frac{h^m}{m!}f^{(m+1)}(\xi)V_m(t)-\sum_{m=0}^{\infty}\frac{h^m}{m!}g^{(m+1)}(\xi)\frac{V_{m+1}(t)}{t}\right]\\ &=-2h\frac{\partial v}{\partial \xi}, \qquad \text{Q. E. D.}\end{aligned}$$

Next we shall show that the supplementary conditions (6) are satisfied by (7). Really, since $V_0(0)=1$ and $V_n(0)=0$, $n=1,2,3,\cdots$ as well as $\left(\frac{V_{n+1}(t)}{t}\right)_0=0$, &c, we see immediately that

$$v_{t=0}=f^{(0)}(\xi)=f(\xi),$$

and

$$\left(\frac{\partial v}{\partial t}\right)_0=\sum_{n=0}^{\infty}\frac{h^n f^{(n)}(\xi)}{n!}V_n'(0)+\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)V_n(0)=g(\xi).$$

Thus the supplementary conditions are all fulfilled.

Furthermore we can show the convergency of (7). Making use of Stirling's formula, when n is sufficiently great, we have for (9)

$$|V_n(t)| \simeq \frac{2^n \cdot n!}{(2n)!}|t|^{2n} \simeq \frac{2^n \cdot n^n \cdot \sqrt{n}}{2^{2n} \cdot n^{2n} \cdot \sqrt{2n}} e^n |t|^{2n} < \frac{1}{\sqrt{2}}\left(\frac{eR^2}{2n}\right)^n, \tag{13}$$

where R denotes the radius of an arbitrary but fixed circle described in t-plane, origin as centre. We have assumed that $f(\xi)$ is regular within a domain D containing $\xi=0$. Hence the inside of $\bar{D}$ can be covered by a finite number of circles, ξ being the centre of one of them, and let the radius be ρ. We have then $\max\limits_{|z-\xi|=\rho} |f(z)| \leq \max\limits_{z \text{ in } D} |f(z)| = M$, so that

$$\left|\frac{f^{(n)}(\xi)}{n!}\right| < \frac{M}{\rho^n} \leq \frac{M}{r^n}, \tag{14}$$

where r denotes the least value among those radii of the covering circles and surely finite (>0 and not infinitely small).[6] Now we can prove the uniform convergency of the first summation in (7) as follows:

From (13) and (14) we get for $|t|<R$ and ξ in D

$$|R_{n_0}(t,\xi)| \equiv \left|\sum_{n=n_0}^{\infty} \frac{h^n}{n!} f^{(n)}(\xi) V_n(t)\right| \leq \frac{M}{\sqrt{2}} \sum_{n=n_0}^{\infty} \left(\frac{ehR^2}{2rn}\right)^n,$$

where the number in the last round bracket may be made $<\varepsilon$ by taking n sufficiently large, say $n \geq n_0$, and thus n_0 can be chosen independently of t and ξ. Consequently

$$|R_{n_0}(t,\xi)| \leq \frac{M}{\sqrt{2}} \sum_{n=n_0} \varepsilon^n = \frac{M\varepsilon^{n_0}}{\sqrt{2}\,(1-\varepsilon)} \qquad (n \geq n_0),$$

and again the last side itself can be made $<$ any prescribed small positive number ε', if we take n_0 sufficiently great, say $n_0 > n_1$. Therefore the first summation in (7) is certainly uniformly convergent. A similar argument could be made about the second summation of (7). Hence the whole expression (7) should be regular in the vicinity of $\xi=0$, $t=0$.

Returning to previous letters, the solution of (2) satisfying the supplementary conditions (5) is given by

$$u = e^{-b\eta} \sum_{n=0}^{\infty} \left(\frac{a}{\mu^2}\right)^n \left\{\frac{f^{(n)}(\xi)}{n!} V_n(\mu\eta) - \frac{F^{(n)}(\xi)+bf^{(n)}(\xi)}{n!} \frac{V_{n+1}(\mu\eta)}{\mu^2\eta}\right\}. \tag{15}$$

It is convenient to take $-\mu^2 = \nu^2 = b^2 - c$, if μ^2 is real negative, and to write (15) in the form

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[6] Or more precisely we may argue as follows. Conceive an inner domain $D_1 \subset D$, which lies wholly inside D, yet almost coincides with it; thus $D_1 = \{z \mid |f(z)| < M, m(D-D_1) < \varepsilon\}$. Let the second inner domain be $D_2 \subset D_1$, whose boundary C_2 is apart ferm C_1, the boundary of D_1, at a distance δ (>0 however small). Now surely D_2 can be covered by a finite number of circles with the property enunciated in text.

$$u = e^{-b\eta}\left\{\sum_{n=0}^{\infty}\left(-\frac{a}{\nu^2}\right)^n \frac{f^{(n)}(\xi)}{\underline{|n}}\, W_n(\nu\eta) + \sum_{n=0}^{\infty}\left(-\frac{a}{\nu^2}\right)^n \frac{F^{(n)}(\xi)+bf^{(n)}(\xi)}{\underline{|n}}\,\frac{W_{n+1}(\nu\eta)}{\nu^2\eta}\right\}, \tag{16}$$

where

$$W_n(\tau) = V_n(i\tau) = \frac{2^n\cdot\tau^{2n}}{\underline{|2n}}\sum_{m=0}^{\infty}\frac{\underline{|m+n}}{\underline{|m}\;\underline{|2m+2n}}\,\tau^{2m}. \tag{17}$$

Lastly, if $b^2 = c$ in (3), so that $\mu = 0$, we see that

$$\lim_{\mu\to 0}\frac{V_n(\mu\eta)}{\mu^{2n}} = \frac{(-1)^n\cdot 2^n\cdot\underline{|n}}{\underline{|2n}}\,\eta^{2n},$$

by (9). Substituting this value in (15), we obtain, as the solution of (2) in the case $\mu = 0$,

$$u = e^{-b\eta}\left[\sum_{n=0}^{\infty}\frac{(-2a)^n}{\underline{|2n}} f^{(n)}(\xi)\eta^{2n} + \sum_{n=0}^{\infty}\frac{(-2a)^n}{\underline{|2n+1}}\,\{F^{(n)}(\xi)+bf^{(n)}(\xi)\}\;\eta^{2n+1}\right]. \tag{18}$$

Example 1. $\dfrac{\partial^2 u}{\partial\eta^2} + 2\dfrac{\partial u}{\partial\xi} + 2\dfrac{\partial u}{\partial\eta} + cu = 0$, where $c = 2$ or 1 or 0, the supplementary conditions being $u = f(\xi)$, $\dfrac{\partial u}{\partial\eta} = F(\xi)$, when $\eta = 0$.

(i) If $c = 2$, we have $\mu^2 = c - b^2 = 1$. Hence, setting $a = b = \mu^2 = 1$ in (15), we obtain the general solution

$$u = e^{-\eta}\left\{\sum_{n=0}^{\infty}\frac{f^{(n)}(\xi)}{\underline{|n}}\,V_n(\eta) - \frac{f^{(n)}(\xi)+F^{(n)}(\xi)}{\underline{|n}}\,\frac{V_{n+1}(\eta)}{\eta}\right\},$$

where

$$V_n(\eta) = \frac{(-1)^n\cdot 2^n\cdot\underline{|n}}{\underline{|2n}}\left\{\eta^{2n} - \frac{\eta^{2n+2}}{(2n+1)\cdot 2\cdot\underline{|1}} + \frac{\eta^{2n+4}}{(2n+1)(2n+3)\cdot 2^n\cdot\underline{|2}} - \cdots\right\}.$$

(ii) If $c = 1$, so $\mu^2 = c - b^2 = 0$, and we ought to take (18). Hence

$$u = e^{-\eta}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{\underline{|2n}} f^{(n)}(\xi)\eta^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{\underline{|2n+1}}\{f^{(n)}(\xi)+F^{(n)}(\xi)\}\;\eta^{2n+1}\right].$$

(iii) If $c = 0$, then $\nu^2 = b^2 - c = 1$. So that we get by (16) and (17)

$$u = e^{-\eta}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n f^{(n)}(\xi)}{\underline{|n}}\,W_n(\eta) + \sum_{n=0}^{\infty}(-1)^n\,\frac{f^{(n)}(\xi)+F^{(n)}(\xi)}{\underline{|n}}\,\frac{W_{n+1}(\eta)}{\eta}\right\},$$

where

$$W_n(\eta) = \frac{2^n\cdot\underline{|n}}{\underline{|2n}}\left\{\eta^{2n} + \frac{\eta^{2n+2}}{(2n+1)\cdot 2\cdot\underline{|1}} + \frac{\eta^{2n+4}}{(2n+1)(2n+3)\cdot 2^2\cdot\underline{|2}} + \cdots\right\}.$$

Example 2. $\dfrac{\partial^2 u}{\partial\eta^2} = \dfrac{\partial u}{\partial\xi}$.

This corresponds to the case $a=-\frac{1}{2}$, $b=c=0$ in (2), and evidently bestows the differential equation of conduction of heat in a linear body with infinite length, where η denotes the distance, while ξ means time! Our solution is nothing to do for well known classical thermal differential equations, such as e.g. with the initial condition $u=\varphi(\eta)$ when $\xi=0$, and besides the boundary condition $u=0$ at $\eta=\pm\infty$. Yet it may be interpretted physically as follows: The supplementary conditions (5) express that the temperature at origin $(=u_{\eta=0})$ in any time ξ, as well as the thermal gradient there $\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}$ are confined to be $f(\xi)$ and $F(\xi)$ respectively, both being given functions of time ξ. As its solution under said conditions (the conditions at origin), we obtain from (18)

$$u=\sum_{n=0}^{\infty}\frac{1}{\underline{|2n}}f^{(n)}(\xi)\eta^{2n}+\sum_{n=0}^{\infty}\frac{1}{\underline{|2n+1}}F^{(n)}(\xi)\eta^{2n+1},$$

which gives the temperature at the distance η in any time ξ.

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EVALUATION OF SOME ω_n^2 DISTRIBUTION

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Some numerical computations of ω_n^2-distribution[1] for the cases $n=2$ and 9 were executed. We have selected them, one as extreme and the other as intermediate, in order to grasp the general feature. We had to evaluate

$$\Phi(\omega_n^2)=1-\frac{\sqrt{n}}{\pi}\sum_{\kappa=1}^{p}(-1)^{\kappa-1}\int_{\varsigma_{2\kappa-1}}^{\varsigma_{2\kappa}}\frac{\exp\left\{-\frac{1}{2}n^2\omega_n^2\varsigma\right\}d\varsigma}{\sqrt{(1-)^n P_{n-1}(\varsigma)\,\varsigma}}, \qquad (1)$$

where $p=\frac{n-1}{2}$ or $\frac{n}{2}$ according as $n=$ odd or even, and ς_ν $(\nu=1,2,\cdots)$ are the roots of $P_{n-1}(\varsigma)=\sum_{l=0}^{n-1}(-1)^l\binom{2n-1-l}{l}\varsigma^{n-1-l}=0$, but if $n=\text{even}=2p$ the last ς_{2p} should be reckoned as ∞.

For convenience of calculation, we put $\varsigma=\frac{1}{2}\left[\varsigma_{2\kappa}+\varsigma_{2\kappa-1}+(\varsigma_{2\kappa}-\varsigma_{2\kappa-1})\sin\frac{\pi}{2}t\right]$, so that (1) becomes

$$\Phi(\omega_n^2)=1-\frac{\sqrt{n}}{2}\sum_{\kappa=1}^{p}(-1)^{\kappa-1}\int_{-1}^{1}\frac{\exp\left\{-\frac{1}{2}n^2\omega_n^2\varsigma\right\}dt}{\varsigma\sqrt{\prod_{\nu\neq\kappa}(\varsigma-\varsigma_{2\nu-1})(\varsigma-\varsigma_{2\nu})}}. \qquad (2)$$

and that will do when $n=2p+1$. However if $n=2p$, the last factor of $\prod_{\nu\neq\kappa}$ for every summand in (2) shoul be single $\varsigma_{2p-1}-\varsigma$, and besides in the last summand of (1), we must set $\varsigma=\varsigma_{2p-1}\sec^2\frac{\pi}{2}t$, so that the corresponding summand in (2) becomes

$$(-1)^{p-1}\frac{\sqrt{n}}{2}\int_{-1}^{1}\frac{\exp\{-2p^2\omega_n^2\}\,dt}{\sqrt{\varsigma_{2p-1}}\sqrt{\prod_{\nu=1}^{2p-2}(\varsigma-\varsigma_\nu)}}.$$

Now that every summand takes form $\frac{1}{2}\int_{-1}^{1}f_\kappa(t)\,dt$, after Gauss, we may equalize it to $\sum R_\nu y_\nu$ with $y_\nu=f(t_\nu,\omega_n^2,\kappa)$ and thus we can compute

$$\Phi(\omega_n^2)=\int_0^{\omega_n^2}\varphi(\omega_n^2)\,d\omega_n^2=1-\sqrt{n}\sum_{\kappa=1}^{p}(-1)^{\kappa-1}\sum_{\nu=1}^{5}R_\nu y_\nu(t_\nu,\omega_n^2,\kappa).$$

[1] Cf. Y. Watanabe, On the ω^2 Distribution, this Journal, Vol. 2, pp. 21-30.

Owing to troublesomeness, much with even n, only the cases $n=2$ and $n=9$ were treated, the results of which are given in the following Tables.[2]

Table of $\Phi(\omega_2^2)=\int_0^{\omega_2^2}\varphi(\omega_2^2)\,d\omega_2^2$

ω_2^2	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	$\Phi=0$	.2528	.3329	.3829	.4254	.4640	.4996	.5084	.5630	.5912
0.1	.6117	.6364	.6594	.6806	.7004	.7188	.7358	.7417	.7473	.7801
0.2	.7950	.8069	.8179	.8281	.8377	.8466	.8560	.8579	.8627	.8769
0.3	.8834	.8896	.8952	.9005	.9049	.9103	.9148	.9174	.9199	.9267
0.4	.9302	.9335	.9367	.9396	.9424	.9451	.9477	.9494	.9511	.9545
0.5	.9542	.9561	.9579	.9597	.9613	.9629	.9644	.9658	.9671	.9686
0.6	.9722	.9734	.9745	.9755	.9766	.9776	.9785	.9793	.9801	.9811
0.7	.9819	.9826	.9834	.9840	.9847	.9853	.9859	.9865	.9870	.9876
0.8	.9881	.9886	.9890	.9895	.9899	.9903	.9907	.9911	.9914	.9918
0.9	.9921	.9924	.9927	.9930	.9933	.9936	.9938	.9941	.9943	.9945
ω_2^2	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	∞
Φ	.9947	.9965	.9977	.9984	.9990	.9992	.9995	9997	.9998	1

Table of $\Phi(\omega_9^2)=\int_0^{\omega_9^2}\varphi(\omega_9^2)\,d\omega_9^2$

ω_9^2	00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	$\Phi=0$	.0060	.0114	.0504	.1025	.1562	.2113	.2723	.3226	.3813
0.1	4267	.4713	.5122	.5495	.5835	.6176	.6428	.6691	.6923	.7145
0.2	.7339	.7522	.7690	.7845	.7987	.8120	.8242	.8355	.8459	.8558
0.3	.8608	8727	.8774	.8849	.8919	.8987	.9045	.9102	.9156	.9206
0.4	.9274	.9316	.9356	.9393	.9429	.9462	.9492	.9522	.9549	.9574
0.5	.9599	.9622	.9643	.9663	.9680	.9700	.9715	.9732	.9747	.9761
0.6	.9775	.9787	.9799	.9810	.9820	.9830	.9839	.9848	.9856	.9864
0.7	.9871	.9878	.9885	.9891	.9897	.9902	.9908	.9912	.9917	.9922
0.8	.9926	.9930	.9933	.9937	.9940	.9944	.9947	.9949	.9952	.9955
0.9	.9957	.9959	.9961	.9963	9965	.9967	.9969	.9970	.9973	.9973
ω_9^2	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	∞
Φ	.9975	.9985	.9991	.9995	.9997	.9998	.9999	.9999	1.0000	1

On comparing the present Tables with that for the case $n=\infty$,[3] we see that their difference is not so remarkable, except when the argument ω^2 is rather small. In fact, the values of $\Phi(\omega_2^2)$ are pretty larger than the corresponding $\Phi(\omega_\infty^2)$ for small ω^2, while with large ω^2 the former are slightly smaller than the latter. The two curves intersect nearly at ($\omega^2=0.427$, $\Phi=0.9386$).

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2) About some parts of computations, especially in regard to numerical solution of $P_{n-1}(\varsigma)=0$, the writer owes to his classmates, Nameda and others.

3) Y. Watanabe, loc. cit., p. 30.

Likewise behaves the curve $\Phi(\omega_9^2)$ to $\Phi(\omega_\infty^2)$ also, but much more approaching to it, and the point of intersection being ($\omega^2 = 0.26$, $\Phi = 0.8242$).

Prof. Watanabe indicates in the end of his note loc. cit., that to test with $\Phi(\omega_\infty^2)$ the hypothetical distribution of Japanese male stature obtained there, it shall be somewhat inadequate.

Now, if we use $\Phi(\omega_9^2)$-Table, and test the same datum, we have, indeed, for $\omega_9^2 = 0.0102$, $\Phi(\omega_9^2) = 0.0077$, so that $1-\Phi = 0.9923$ (> 0.05), while for $\omega_\infty^2 = 0.0102$, $1-\Phi(\omega_\infty^2) = 0.9993$. As it holds still alike that the hypothetical distribution is permissive, yet the degree of probability is moderately $P = 0.9923$, less than 0.9993, which is too near 1, and thus the argument becomes more plausible.

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SOME INVESTIGATIONS IN MATHEMATICAL CARTOGRAPHY

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Introduction

Since Tissot's noted works[1] were published, we have been interested in the studies of the distortion theories in mathematical cartography. In recent years König has discussed the distortion theories of conformal projections in his noted book " Mathematische Grundlagen der Höheren Geodäsie und Kartographie ". In this book, he refered to the conformal conical projections and their modifications, under the title of " Allgemeinen Kegelprojektionen "[2].

Indeed we can say that the problems of distortions in cartography are reducible to solve some problems concerning with the partial differential equations. In König's book he treated them as some problems of initial conditions in the theories of partial differential equations.

Considering these problems, a step forward, we desire to get some new types of modifications in conformal conical projections. It is very interesting to solve the distortion problems under the various kinds of conditions, in theoretically or practically. We desire to refer to these kinds of problems in next chances.

I Conformal Projections between Two Surfaces.

In the theories of differential geometry, we know that, if on each of two surfaces, a pair of isometric coordinates u, v and u', v' are chosen, the equation $u+iv=f(u'+iv')$ defines all conformal projections between these surfaces, when the function f is analytic with regard to the variable $u'+iv'$.

In cartography we consider mainly conformal projections between a spheroid, including a sphere as a special case, and a plane. So if we put the isometric coodinates on the spheroid to H and L, we get

$$H=\log\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{\frac{e}{2}}\right]^{3)} \qquad \cdots\cdots\cdots\cdots(1),$$

where B is the geographical latitude on the spheroid and e is its exentricity.

As e is equal to zero on the sphere, we get

$$H = \log\, \tan\left(\frac{\pi}{4} + \frac{B}{2}\right) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2),$$

on the other side, L is the longitude measured from a suitable meridian.

Chosing these isometric coordinates, we get

$$ds_s^2 = N^2 \cos^2 B(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3),$$

where ds_s is the line element on the surface, and N is the radius of transverse curvature at the given point. So N is represented to

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 B}} \text{ [4]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4),$$

where a is the length of the major axis. Of course, in the case of sphere, as e is equal to zero, N is equal to the radius R; and ds_s^2 is represented to

$$ds_s^2 = R^2 \cos^2 B(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5).$$

Now it is clear that the plane rectangular coordinates x and y are isometric, so we conclude that the general conformal projections between a sheroid and a plane is represented to

$$z = x + iy = f(H + iL) = x(H, L) + y(H, L) \text{ [5]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6),$$

where z is an analytic function of $H + iL$.

So the line element ds_p on the plane is

$$ds_p^2 = \lambda^2(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7),$$

where

$$\lambda^2 = \left(\frac{\partial x}{\partial H}\right)^2 + \left(\frac{\partial x}{\partial L}\right)^2 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(8);$$

then the distortion m is given by

$$m^2 = \frac{ds_p^2}{ds_s^2} = \frac{\lambda^2}{N^2 \cos^2 B} \text{ [6]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(9).$$

All the distortion problems concerning with the conformal projections are based on the equation (9).

II Conformal Conical Projections and Their Modifications.

According to the König's symbols, we put $H + iL$ to M and introduce Λ from the expression

$$\Lambda = \kappa e^{-\alpha M} \text{ [7]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(10)$$

where α and k are pure constants. If we put the real and imaginary parts of Λ to x and y respectively, we get the following conformal projection

$$x = ke^{-\alpha H} \cos(\alpha L) \qquad y = -ke^{-\alpha H} \sin(\alpha L) \qquad \cdots\cdots\cdots(11).$$

Calculating from (11), we get $m^2 = \dfrac{k^2\alpha^2 e^{-2\alpha H}}{N^2 \cos^2 B}$ $\cdots\cdots\cdots(12)$.

Then we can get $\qquad k = N_0 \cot B_0 e^{\alpha M_0}$ $\cdots\cdots\cdots(13)$.

and $\qquad \alpha = \sin B_0$ $\cdots\cdots\cdots(14)$,

respectively, for m shall be unity at the given latitude B_0, and H_0, N_0, mean the values of N and H at the given latitude B_0. It is the principles of the theories of ordinary conformal conical projections.

König desired to extend the above projections, by introducing Λ_A such as

$$\Lambda_A = \frac{a\Lambda + b}{c\Lambda + d} \text{ [8]} \qquad \cdots\cdots\cdots(15),$$

where a, b, c, d, are any constants, and he defined Λ_A to " Allgemeinen Kegel-projektionen ". By introducing Λ_A he got the various kinds of projections. So not only whose distortion m is equal to unity at the central point $(B_0, L_0,)$, but they are satisfied by the various properties of conditions In these projections, generally, it is impossible to make m unity along the definite parallel B_0, at any longitude L. Because of the facts, it is not suitable to say that these projections to conical projections, in strict sense; so we called them the modifications of conical projections. Adding to the conditions that m is equal to unity, we desire to all the values of $\dfrac{\partial m}{\partial H}$, $\dfrac{\partial m}{\partial L}$, $\dfrac{\partial^2 m}{\partial H^2}$, $\dfrac{\partial^2 m}{\partial H^2}$, $\dfrac{\partial^2 m}{\partial L^2}$ etc. are equal to zero at the central point.

So we extend the expression (10) to the form

$$\Lambda = e^{f(M)} = e^{p+iq} \qquad \cdots\cdots\cdots(16),$$

such that the function $f(M)$ is an anatytic function of M. Then we desire to decide the forms of f, so as to all the above conditions are satisfied. After the deciding of the function f, we put the real and imaginary parts of (16) to x and y, and get the conformal projection

$$x = e^p \cos q \qquad y = e^p \sin q \qquad \cdots\cdots\cdots(17).$$

From (9) and (17), we get

$$\mu = m^2 = \frac{2e^{2p}}{N^2 \cos B^2}(R^2 + S^2) \qquad \cdots\cdots\cdots(18),$$

where R and S in (18) mean that

$$R = \frac{\partial p}{\partial H} = \frac{\partial q}{\partial L}, \quad S = \frac{\partial p}{\partial L} = -\frac{\partial q}{\partial H}, \qquad \cdots\cdots\cdots(19).$$

From (18), calculating $\frac{\partial\mu}{\partial H}$ etc. we get the conditions of p and q, which are satisfied by the following equations

$$\left(\frac{\partial H}{\partial\mu}\right)_0=\left(\frac{\partial\mu}{\partial L}\right)_0=\left(\frac{\partial^2\mu}{\partial H^2}\right)_0=\left(\frac{\partial^2\mu}{\partial H\partial L}\right)_0=\left(\frac{\partial^2\mu}{\partial L^2}\right)_0=0 \quad\cdots\cdots\cdots\cdots(20),$$

where $\left(\frac{\partial\mu}{\partial H}\right)_0$ etc. mean the values of them at the central point.

So $f(M)$ is an analytic function of M, we see that the followings hold,

$$\frac{\partial R}{\partial H}=-\frac{\partial S}{\partial L},\ \frac{\partial R}{\partial L}=\frac{\partial S}{\partial H},\ \frac{\partial^2 R}{\partial H^2}=-\frac{\partial^2 R}{\partial L^2}=\frac{\partial^2 S}{\partial H\partial L},\ \frac{\partial^2 S}{\partial H^2}=-\frac{\partial^2 S}{\partial L^2}=-\frac{\partial^2 R}{\partial H\partial L} \quad\cdots\cdots\cdots\cdots(21).$$

Using the conditions of (21), we get the followings by differentiating of (18),

$$\begin{aligned}\left(\frac{\partial\mu}{\partial H}\right)_0&=\frac{4e^{2p_0}}{N^2\cos^2 B_0}\left[R\frac{\partial R}{\partial H}+S\frac{\partial S}{\partial H}+(S^2+R^2)R+(S^2+R^2)\sin B\right]_0\\ \left(\frac{\partial\mu}{\partial L}\right)_0&=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[R\frac{\partial S}{\partial H}-S\frac{\partial R}{\partial H}+(S^2+R^2)S\right]_0\end{aligned} \quad\cdots\cdots\cdots\cdots(22),$$

where N_0, B_0, P_0, mean the values of N, B, P, at the central point; in the calculation of the above the effects of the exentricity were neglected, for it is very small.

From (20) and (22), we get the following equations

$$\left.\begin{aligned}&\left[R\frac{\partial R}{\partial H}+S\frac{\partial S}{\partial H}+(S^2+R^2)R+(S^2+R^2)\sin B\right]_0=0\\ &\left[R\frac{\partial S}{\partial H}-S\frac{\partial R}{\partial H}+(S^2+R^2)S\right]_0=0\end{aligned}\right\}\cdots\cdots\cdots\cdots(23).$$

After the differentation of (22), we get the followings by using of the results of (20) and (23),

$$\left.\begin{aligned}\left(\frac{\partial^2\mu}{\partial H^2}\right)_0&=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[\left(\frac{\partial R}{\partial H}\right)^2+R\left(\frac{\partial^2 R}{\partial H^2}\right)+\left(\frac{\partial S}{\partial H}\right)^2+S\frac{\partial^2 S}{\partial H^2}+S^2\frac{\partial R}{\partial H}+2SR\frac{\partial S}{\partial H}\right.\\ &\quad\left.+3R^2\frac{\partial R}{\partial H}+2S\sin B\frac{\partial S}{\partial H}+2R\sin B\frac{\partial R}{\partial H}+\cos^2 B(S^2+R^2)\right]_0=0\\ \left(\frac{\partial^2\mu}{\partial H\partial L_0}\right)_0&=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[R\frac{\partial^2 S}{\partial H^2}-S\frac{\partial^2 R}{\partial H^2}+S^2\frac{\partial S}{\partial H}-2RS\frac{\partial R}{\partial H}+3R^2\frac{\partial R}{\partial H}\right.\\ &\quad\left.+2R\sin B\frac{\partial S}{\partial H}-2S\sin B\frac{\partial R}{\partial H}\right]_0=0\\ \left(\frac{\partial^2\mu}{\partial L^2}\right)_0&=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[\left(\frac{\partial S}{\partial H}\right)^2+R\left(\frac{\partial^2 R}{\partial H^2}\right)+S\frac{\partial^2 S}{\partial H^2}+\left(\frac{\partial R}{\partial H}\right)^2-R^2\left(\frac{\partial R}{\partial H}\right)\right.\\ &\quad\left.+2RS\frac{\partial S}{\partial H}-3S^2\frac{\partial R}{\partial H}\right]_0=0\end{aligned}\right\}(24).$$

If we assume that the function $f(M)=p+iq$ is developable in Taylor's series in the neighbourhood of the point (0, 0), it is possible to put it to

$$f(M)=\alpha_0+\alpha_1 M+\alpha_2 M^2+\alpha_3 M^3+\cdots \qquad \cdots\cdots(25),$$

Assuming all $\alpha_i\,(i\geq 4)$ are zero, we can put p, and q, to the form

$$\left.\begin{aligned} p&=\alpha_0+\alpha_1 H+\alpha_2(H^2-L^2)+\alpha_3(H^3-3HL^2)\\ q&=\alpha_1 L+2\alpha_2 HL+\alpha_3(3H^2L-L^3)\end{aligned}\right\} \qquad \cdots\cdots(26).$$

Then we get from (19) and (21),

$$R=\alpha_1+2\alpha_2 H+3\alpha_3(H^2-L^2) \quad S=-2\alpha_2 L-6\alpha_3 HL \qquad \cdots\cdots(27),$$

and $$\frac{\partial R}{\partial H}=2\alpha_2+6\alpha_3 H,\ \frac{\partial S}{\partial H}=-6\alpha_3 L,\ \frac{\partial^2 R}{\partial H^2}=6\alpha_3,\ \frac{\partial^2 S}{\partial H^2}=0 \qquad \cdots\cdots(28).$$

Putting the value of L at the central point to zero, we see that the values of S, $\frac{\partial S}{\partial H}$ $\frac{\partial^2 S}{\partial H^2}$ etc. are equal to zero at the point, and the equations (23), (24) are reducible to the form

$$\left.\begin{aligned} &R_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^3+R_0^2\sin B_0=0\\ &\left(\frac{\partial R}{\partial H}\right)_0^2+R_0^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0+3R_0^2\left(\frac{\partial R}{\partial H}\right)_0+2R_0\sin B_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^2\cos^2 B_0=0\\ &\left(\frac{\partial R}{\partial H}\right)_0^2+R_0\left(\frac{\partial^2 R}{\partial H^2}\right)_0-R_0^2\left(\frac{\partial R}{\partial H}\right)_0=0\end{aligned}\right\} \cdots(29).$$

Other equations of (23) and (24) all vanish at the central point naturally. Solving the equations of (27), we can decide the values of R_0, $\left(\frac{\partial R}{\partial H_0}\right)_0$, $\left(\frac{\partial^2 R}{\partial H^2}\right)_0$, and from (27) and (28), we can get the values of α_1, α_2, and α_3.

For simplicity, if we take the central point on the equator and put B_0 to zero, the above equations (29) are reduced to the form,

$$R_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^3=0 \qquad \left(\frac{\partial R}{\partial H}\right)_0+R_0^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0+3R_0^2\left(\frac{\partial R}{\partial H}\right)_0+R_0^2=0$$

$$\left(\frac{\partial R}{\partial H}\right)_0^2+R^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0-R_0^2\left(\frac{\partial R}{\partial H}\right)_0^2=0 \qquad \cdots\cdots(30).$$

Solving the equation (30), we get

$$R_0=\frac{1}{2} \qquad \left(\frac{\partial R}{\partial H}\right)_0=-\frac{1}{4} \qquad \left(\frac{\partial^2 R}{\partial H^2}\right)_0=\frac{1}{4},$$

and $$R_0=-\frac{1}{2} \qquad \left(\frac{\partial R}{\partial H}\right)_0=-\frac{1}{4} \qquad \left(\frac{\partial^2 R}{\partial H^2}\right)_0=-\frac{1}{4} \qquad \cdots\cdots(31),$$

from these results and (28), (29), we get

$$\alpha_1=\frac{1}{2} \qquad \alpha_2=-\frac{1}{8} \qquad \alpha_4=\frac{1}{24}$$

$$\alpha_1 = -\frac{1}{2} \qquad \alpha_2 = -\frac{1}{8} \qquad \alpha_4 = -\frac{1}{24} \qquad \cdots\cdots\cdots\cdots(32).$$

In general, giving any value of B_0, it is difficult to solve the values of $\alpha_1, \alpha_2, \alpha_3$, so simply. But we can solve these equations by suitable algebraic method. In this discussion we did not refer to the condition that m is equal to unity at the central point; for e^{α_0} is a constant, so α_0 may be decided similary the method of (14).

At the end of this paper, we shall refer to the case when $\alpha_i s$ take any complex values.

Indeed, if we put α_i to a pure imaginary $i\beta_i$, H and L were interchanged in (26), so the conical projection is said a transeverse projection.

So in general if $\alpha_i s$ take any complex values we can say that it is the case of the oblique projection. Generally speaking, it is difficult to treat the case of oblique projections, gemetrically on the spheroid; but by means of introducing of complex coefficients, we can consider them very easily.

These conditions which are considered now, are nothing but some initial conditions at the central point. But it is understood, that the conditions which are requested in the conical projection with two standard paralleles are so to speak some primitive boundary conditions. So it is expected to get some useful projections by the extensions of these methods.

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(8) see, page 360 of (2).

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CONTENTS

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