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# SOME INVESTIGATIONS IN MATHEMATICAL CARTOGRAPHY

Takaharu MARUYAMA

*Mathematical Institute, Gakugei College, Tokushima University.*

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## Introduction

Since Tissot's noted works[1] were published, we have been interested in the studies of the distortion theories in mathematical cartography. In recent years König has discussed the distortion theories of conformal projections in his noted book " Mathematische Grundlagen der Höheren Geodäsie und Kartographie ". In this book, he refered to the conformal conical projections and their modifications, under the title of " Allgemeinen Kegelprojektionen "[2].

Indeed we can say that the problems of distortions in cartography are reducible to solve some problems concerning with the partial differential equations. In König's book he treated them as some problems of initial conditions in the theories of partial differential equations.

Considering these problems, a step forward, we desire to get some new types of modifications in conformal conical projections. It is very interesting to solve the distortion problems under the various kinds of conditions, in theoretically or practically. We desire to refer to these kinds of problems in next chances.

## I Conformal Projections between Two Surfaces.

In the theories of differential geometry, we know that, if on each of two surfaces, a pair of isometric coordinates $u, v$ and $u', v'$ are chosen, the equation $u+iv=f(u'+iv')$ defines all conformal projections between these surfaces, when the function $f$ is analytic with regard to the variable $u'+iv'$.

In cartography we consider mainly conformal projections between a spheroid, including a sphere as a special case, and a plane. So if we put the isometric coodinates on the spheroid to $H$ and $L$, we get

$$H=\log\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{\frac{e}{2}}\right]^{3)} \qquad \cdots\cdots\cdots\cdots(1),$$

where $B$ is the geographical latitude on the spheroid and $e$ is its exentricity.

As $e$ is equal to zero on the sphere, we get

$$H = \log \tan\left(\frac{\pi}{4} + \frac{B}{2}\right) \qquad \cdots\cdots\cdots\cdots\cdots\cdots (2),$$

on the other side, $L$ is the longitude measured from a suitable meridian.

Chosing these isometric coordinates, we get

$$ds_s^2 = N^2 \cos^2 B(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots (3),$$

where $ds_s$ is the line element on the surface, and $N$ is the radius of transverse curvature at the given point. So $N$ is represented to

$$N = \frac{a}{\sqrt{1-e^2 \sin^2 B}} \text{ [4]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (4),$$

where $a$ is the length of the major axis. Of course, in the case of sphere, as $e$ is equal to zero, $N$ is equal to the radius $R$; and $ds_s^2$ is represented to

$$ds_s^2 = R^2 \cos^2 B(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots (5).$$

Now it is clear that the plane rectangular coordinates $x$ and $y$ are isometric, so we conclude that the general conformal projections between a sheroid and a plane is represented to

$$z = x + iy = f(H + iL) = x(H, L) + y(H, L) \text{ [5]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (6),$$

where $z$ is an analytic function of $H + iL$.

So the line element $ds_p$ on the plane is

$$ds_p^2 = \lambda^2(dH^2 + dL^2) \qquad \cdots\cdots\cdots\cdots\cdots\cdots (7),$$

where

$$\lambda^2 = \left(\frac{\partial x}{\partial H}\right)^2 + \left(\frac{\partial x}{\partial L}\right)^2 \qquad \cdots\cdots\cdots\cdots\cdots\cdots (8);$$

then the distortion $m$ is given by

$$m^2 = \frac{ds_p^2}{ds_s^2} = \frac{\lambda^2}{N^2 \cos^2 B} \text{ [6]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (9).$$

All the distortion problems concerning with the conformal projections are based on the equation (9).

## II Conformal Conical Projections and Their Modifications.

According to the König's symbols, we put $H + iL$ to $M$ and introduce $\Lambda$ from the expression

$$\Lambda = \kappa e^{-\alpha M} \text{ [7]} \qquad \cdots\cdots\cdots\cdots\cdots\cdots (10)$$

where $\alpha$ and $k$ are pure constants. If we put the real and imaginary parts of $\Lambda$ to $x$ and $y$ respectively, we get the following conformal projection

$$x = ke^{-\alpha H} \cos(\alpha L) \qquad y = -ke^{-\alpha H} \sin(\alpha L) \qquad \cdots\cdots\cdots(11).$$

Calculating from (11), we get $m^2 = \dfrac{k^2\alpha^2 e^{-2\alpha H}}{N^2 \cos^2 B}$ ..................(12).

Then we can get $k = N_0 \cot B_0 e^{\alpha M_0}$ ..................(13).

and $\alpha = \sin B_0$ ..................(14),

respectively, for $m$ shall be unity at the given latitude $B_0$, and $H_0$, $N_0$, mean the values of $N$ and $H$ at the given latitude $B_0$. It is the principles of the theories of ordinary conformal conical projections.

König desired to extend the above projections, by introducing $\Lambda_A$ such as

$$\Lambda_A = \frac{a\Lambda + b}{c\Lambda + d} \text{ [8)]} \qquad \cdots\cdots\cdots(15),$$

where $a$, $b$, $c$, $d$, are any constants, and he defined $\Lambda_A$ to " Allgemeinen Kegelprojektionen ". By introducing $\Lambda_A$ he got the various kinds of projections. So not only whose distortion $m$ is equal to unity at the central point $(B_0, L_0,)$, but they are satisfied by the various properties of conditions In these projections, generally, it is impossible to make $m$ unity along the definite parallel $B_0$, at any longitude $L$. Because of the facts, it is not suitable to say that these projections to conical projections, in strict sense; so we called them the modifications of conical projections. Adding to the conditions that $m$ is equal to unity, we desire to all the values of $\dfrac{\partial m}{\partial H}$, $\dfrac{\partial m}{\partial L}$, $\dfrac{\partial^2 m}{\partial H^2}$, $\dfrac{\partial^2 m}{\partial H^2}$, $\dfrac{\partial^2 m}{\partial L^2}$ etc. are equal to zero at the central point.

So we extend the expression (10) to the form

$$\Lambda = e^{f(M)} = e^{p+iq} \qquad \cdots\cdots\cdots(16),$$

such that the function $f(M)$ is an anatytic function of $M$. Then we desire to decide the forms of $f$, so as to all the above conditions are satisfied. After the deciding of the function $f$, we put the real and imaginary parts of (16) to $x$ and $y$, and get the conformal projection

$$x = e^p \cos q \qquad y = e^p \sin q \qquad \cdots\cdots\cdots(17).$$

From (9) and (17), we get

$$\mu = m^2 = \frac{2e^{2p}}{N^2 \cos B^2}(R^2 + S^2) \qquad \cdots\cdots\cdots(18),$$

where $R$ and $S$ in (18) mean that

$$R = \frac{\partial p}{\partial H} = \frac{\partial q}{\partial L}, \quad S = \frac{\partial p}{\partial L} = -\frac{\partial q}{\partial H}, \qquad \cdots\cdots\cdots(19).$$

From (18), calculating $\frac{\partial\mu}{\partial H}$ etc. we get the conditions of $p$ and $q$, which are satisfied by the following equations

$$\left(\frac{\partial H}{\partial\mu}\right)_0=\left(\frac{\partial\mu}{\partial L}\right)_0=\left(\frac{\partial^2\mu}{\partial H^2}\right)_0=\left(\frac{\partial^2\mu}{\partial H\partial L}\right)_0=\left(\frac{\partial^2\mu}{\partial L^2}\right)_0=0 \quad\cdots\cdots\cdots\cdots(20),$$

where $\left(\frac{\partial\mu}{\partial H}\right)_0$ etc. mean the values of them at the central point.

So $f(M)$ is an analytic function of $M$, we see that the followings hold,

$$\frac{\partial R}{\partial H}=-\frac{\partial S}{\partial L},\ \frac{\partial R}{\partial L}=\frac{\partial S}{\partial H},\ \frac{\partial^2 R}{\partial H^2}=-\frac{\partial^2 R}{\partial L^2}=\frac{\partial^2 S}{\partial H\partial L},\ \frac{\partial^2 S}{\partial H^2}=-\frac{\partial^2 S}{\partial L^2}=-\frac{\partial^2 R}{\partial H\partial L} \quad\cdots\cdots\cdots\cdots(21).$$

Using the conditions of (21), we get the followings by differentiating of (18),

$$\left(\frac{\partial\mu}{\partial H}\right)_0=\frac{4e^{2p_0}}{N^2\cos^2 B_0}\left[R\frac{\partial R}{\partial H}+S\frac{\partial S}{\partial H}+(S^2+R^2)R+(S^2+R^2)\sin B\right]_0$$

$$\left(\frac{\partial\mu}{\partial L}\right)_0=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[R\frac{\partial S}{\partial H}-S\frac{\partial R}{\partial H}+(S^2+R^2)S\right]_0 \quad\cdots\cdots\cdots\cdots(22),$$

where $N_0$, $B_0$, $P_0$, mean the values of $N$, $B$, $P$, at the central point; in the calculation of the above the effects of the exentricity were neglected, for it is very small.

From (20) and (22), we get the following equations

$$\left.\begin{aligned}&\left[R\frac{\partial R}{\partial H}+S\frac{\partial S}{\partial H}+(S^2+R^2)R+(S^2+R^2)\sin B\right]_0=0\\&\left[R\frac{\partial S}{\partial H}-S\frac{\partial R}{\partial H}+(S^2+R^2)S\right]_0=0\end{aligned}\right\}\cdots\cdots\cdots\cdots(23).$$

After the differentation of (22), we get the followings by using of the results of (20) and (23),

$$\left.\begin{aligned}&\left(\frac{\partial^2\mu}{\partial H^2}\right)_0=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[\left(\frac{\partial R}{\partial H}\right)^2+R\left(\frac{\partial^2 R}{\partial H^2}\right)+\left(\frac{\partial S}{\partial H}\right)^2+S\frac{\partial^2 S}{\partial H^2}+S^2\frac{\partial R}{\partial H}+2SR\frac{\partial S}{\partial H}\right.\\&\qquad\left.+3R^2\frac{\partial R}{\partial H}+2S\sin B\frac{\partial S}{\partial H}+2R\sin B\frac{\partial R}{\partial H}+\cos^2 B(S^2+R^2)\right]_0=0\\&\left(\frac{\partial^2\mu}{\partial H\partial L_0}\right)_0=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[R\frac{\partial^2 S}{\partial H^2}-S\frac{\partial^2 R}{\partial H^2}+S^2\frac{\partial S}{\partial H}-2RS\frac{\partial R}{\partial H}+3R^2\frac{\partial R}{\partial H}\right.\\&\qquad\left.+2R\sin B\frac{\partial S}{\partial H}-2S\sin B\frac{\partial R}{\partial H}\right]_0=0\\&\left(\frac{\partial^2\mu}{\partial L^2}\right)_0=\frac{4e^{2p_0}}{N_0^2\cos^2 B_0}\left[\left(\frac{\partial S}{\partial H}\right)^2+R\left(\frac{\partial^2 R}{\partial H^2}\right)+S\frac{\partial^2 S}{\partial H^2}+\left(\frac{\partial R}{\partial H}\right)^2-R^2\left(\frac{\partial R}{\partial H}\right)\right.\\&\qquad\left.+2RS\frac{\partial S}{\partial H}-3S^2\frac{\partial R}{\partial H}\right]_0=0\end{aligned}\right\}(24).$$

If we assume that the function $f(M)=p+iq$ is developable in Taylor's series in the neighbourhood of the point (0, 0), it is possible to put it to

$$f(M)=\alpha_0+\alpha_1 M+\alpha_2 M^2+\alpha_3 M^3+\cdots \qquad \cdots\cdots(25),$$

Assuming all $\alpha_i\,(i\geqq 4)$ are zero, we can put $p$, and $q$, to the form

$$\left.\begin{aligned} p&=\alpha_0+\alpha_1 H+\alpha_2(H^2-L^2)+\alpha_3(H^3-3HL^2)\\ q&=\alpha_1 L+2\alpha_2 HL+\alpha_3(3H^2L-L^3)\end{aligned}\right\} \qquad \cdots\cdots(26).$$

Then we get from (19) and (21),

$$R=\alpha_1+2\alpha_2 H+3\alpha_3(H^2-L^2) \quad S=-2\alpha_2 L-6\alpha_3 HL \qquad \cdots\cdots(27),$$

and $$\frac{\partial R}{\partial H}=2\alpha_2+6\alpha_3 H,\ \frac{\partial S}{\partial H}=-6\alpha_3 L,\ \frac{\partial^2 R}{\partial H^2}=6\alpha_3,\ \frac{\partial^2 S}{\partial H^2}=0 \qquad \cdots\cdots(28).$$

Putting the value of $L$ at the central point to zero, we see that the values of $S$, $\frac{\partial S}{\partial H}\,\frac{\partial^2 S}{\partial H^2}$ etc. are equal to zero at the point, and the equations (23), (24) are reducible to the form

$$\left.\begin{aligned} &R_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^3+R_0^2\sin B_0=0\\ &\left(\frac{\partial R}{\partial H}\right)_0^2+R_0^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0+3R_0^2\left(\frac{\partial R}{\partial H}\right)_0+2R_0\sin B_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^2\cos^2 B_0=0\\ &\left(\frac{\partial R}{\partial H}\right)_0^2+R_0\left(\frac{\partial^2 R}{\partial H^2}\right)_0-R_0^2\left(\frac{\partial R}{\partial H}\right)_0=0\end{aligned}\right\} \cdots(29).$$

Other equations of (23) and (24) all vanish at the central point naturally. Solving the equations of (27), we can decide the values of $R_0$, $\left(\frac{\partial R}{\partial H_0}\right)_0$, $\left(\frac{\partial^2 R}{\partial H^2}\right)_0$, and from (27) and (28), we can get the values of $\alpha_1$, $\alpha_2$, and $\alpha_3$.

For simplicity, if we take the central point on the equator and put $B_0$ to zero, the above equations (29) are reduced to the form,

$$R_0\left(\frac{\partial R}{\partial H}\right)_0+R_0^3=0 \qquad \left(\frac{\partial R}{\partial H}\right)_0+R_0^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0+3R_0^2\left(\frac{\partial R}{\partial H}\right)_0+R_0^2=0$$

$$\left(\frac{\partial R}{\partial H}\right)_0^2+R^2\left(\frac{\partial^2 R}{\partial H^2}\right)_0-R_0^2\left(\frac{\partial R}{\partial H}\right)_0^2=0 \qquad \cdots\cdots(30).$$

Solving the equation (30), we get

$$R_0=\frac{1}{2} \qquad \left(\frac{\partial R}{\partial H}\right)_0=-\frac{1}{4} \qquad \left(\frac{\partial^2 R}{\partial H^2}\right)_0=\frac{1}{4},$$

and $$R_0=-\frac{1}{2} \qquad \left(\frac{\partial R}{\partial H}\right)_0=-\frac{1}{4} \qquad \left(\frac{\partial^2 R}{\partial H^2}\right)_0=-\frac{1}{4} \qquad \cdots\cdots(31),$$

from these results and (28), (29), we get

$$\alpha_1=\frac{1}{2} \qquad \alpha_2=-\frac{1}{8} \qquad \alpha_4=\frac{1}{24}$$

$$\alpha_1 = -\frac{1}{2} \qquad \alpha_2 = -\frac{1}{8} \qquad \alpha_4 = -\frac{1}{24} \qquad \cdots\cdots\cdots\cdots(32).$$

In general, giving any value of $B_0$, it is difficult to solve the values of $\alpha_1, \alpha_2, \alpha_3$, so simply. But we can solve these equations by suitable algebraic method. In this discussion we did not refer to the condition that $m$ is equal to unity at the central point; for $e^{\alpha_0}$ is a constant, so $\alpha_0$ may be decided similary the method of (14).

At the end of this paper, we shall refer to the case when $\alpha_i s$ take any complex values.

Indeed, if we put $\alpha_i$ to a pure imaginary $i\beta_i$, $H$ and $L$ were interchanged in (26), so the conical projection is said a transeverse projection.

So in general if $\alpha_i s$ take any complex values we can say that it is the case of the oblique projection. Generally speaking, it is difficult to treat the case of oblique projections, gemetrically on the spheroid; but by means of introducing of complex coefficients, we can consider them very easily.

These conditions which are considered now, are nothing but some initial conditions at the central point. But it is understood, that the conditions which are requested in the conical projection with two standard paralleles are so to speak some primitive boundary conditions. So it is expected to get some useful projections by the extensions of these methods.

## Bibliography

(1) Tissot; Memoire sur la représentation des surfaces géographique et des projections des cartes, Paris, 1881.

(2) R. König und K. H. Weise; Mathematische Grundlagen der Höheren Geodäsie und Kartographie. Springer. 1951.

(3) Driencourt, L. et J. Laborde; Traité des projections des cartes géographique, Paris. 1932.

(4) L. Eisenhart; An introduction to differential geometry with use of the tensor calculas; Princeton. 1947.

## References

(1) see, the bibliography (1).
(2) see, page 369 of (2).
(3) see, page 72 of (2).
(4) see, page 10 of (2).
(5) see, page 101 of (4).
(6) see, page 231 of (2).
(7) see, page 359 of (2).
(8) see, page 360 of (2).