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ON THE PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH CONSTANT COEFFICIENTS

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We consider the linear homogeneous partial differential equation of second order with constant coefficients[1)]

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+H\frac{\partial u}{\partial x}+K\frac{\partial u}{\partial y}+Lu=0. \qquad (1)$$

In particular, when it is of parabolic type, i. e. if $B^2-4AC=0$, then by the familiar transformation $\xi=y-\alpha x$, $\eta=y$, α being the equal root of the equation $A\gamma^2-B\gamma+C=0$, we have

$$\frac{\partial^2 u}{\partial\eta^2}+2a\frac{\partial u}{\partial\xi}+2b\frac{\partial u}{\partial\eta}+cu=0, \qquad (2)$$

where $a=\dfrac{2AK-BH}{B^2}$, $b=\dfrac{K}{2C}$, $c=\dfrac{L}{C}$.[2)] Or, on writing

$$b^2-c=-\mu^2(\neq 0),^{3)}\quad \eta=\frac{t}{\mu},\quad \frac{a}{\mu^2}=h,\quad \frac{b}{\mu}=k \quad\text{and}\quad u=ve^{-kt}, \qquad (3)$$

we obtain

$$\frac{\partial^2 v}{\partial t^2}+2h\frac{\partial v}{\partial\xi}+v=0. \qquad (4)$$

Let supplementary conditions be such that, in regard to (2)

$$u=f(\xi),\quad \frac{\partial u}{\partial\eta}=F(\xi),\qquad \text{when}\quad \eta=0, \qquad (5)$$

which become for (4)

$$v=f(\xi),\quad \frac{\partial v}{\partial t}=\frac{F(\xi)+bf(\xi)}{\mu}\equiv g(\xi),\qquad \text{when}\quad t=0, \qquad (6)$$

where it is assumed that $f(\xi)$ as well as $F(\xi)$ (and so also $g(\xi)$) all permit

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1) The constants as well as the variables are supposed to be complex in general.

2) Here we have assumed as $BC\neq0$; but if one of B and C (or A) vanishes, so also the other must vanish because of $B^2=4AC$, and then the original equation (1) already has the form (2) or alike.

3) The case $\mu=0$ i. e. $b^2=c$ shall be considered later on.

Taylor's expansion in the vicinity of $\xi=0$.[4] If the solution of (4) satisfying (6) could be found, it should contain two arbitrarily chosen functions $f(\xi)$ and $g(\xi)$, so that it might be regarded as the general solution of (1), if the letters be put back.

For our purpose, we think the equation of hyperbolic type

$$\frac{\partial^2 v}{\partial t^2}-\varepsilon^2\frac{\partial^2 v}{\partial \xi^2}+2h\frac{\partial v}{\partial \xi}+v=0,$$

whose solution is obtainable by Riemann's method with Bessel function. Now making $\varepsilon\to 0$, we get, as the required solution, superficially

$$v=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)V_n(t)-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)\frac{V_{n+1}(t)}{t}, \tag{7}$$

where $V_n(t)$ stands for

$$V_n(t)=\left[\frac{d^n}{d\zeta^n}\cos(\sqrt{1+2\zeta}\,t)\right]_{\zeta=0}, \tag{8}$$

and consequently

$$V_0(t)=\cos t,\ V_1(t)=-t\sin t,\ V_2(t)=-t_2\cos t+t\sin t,$$
$$V_3(t)=3t^2\cos t+(t^3-3t)\sin t,\ V_4(t)=(t^4-15t^2)\cos t-(6t^3-15t)\sin t,$$
$$V_5(t)=-(10t^4-105t^2)\cos t-(t^5-45t^3+1054t)\sin t,\ \text{and so on.}$$

The arguement t being complex in general, if $t=\theta+\tau\sqrt{-1}$ (θ, τ real), so it must be understood that

$$\cos t=\cos\theta\cdot\cosh\tau-i\sin\theta\sinh\tau,\ \sin t=\sin\theta\cdot\cosh\tau+i\cos\theta\cdot\sinh\tau.$$

Or expanding $V_n(t)$ in power series, we get

$$V_n(t)=\sum_{l=n}^{\infty}\frac{(-1)^l t^{2l}}{(2l)!}\left[\frac{d^n}{d\zeta^n}(1+2\zeta)^l\right]_{\zeta=0}=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}\lfloor m+n}{\lfloor m\ \lfloor 2m+2n}2^n t^{2m+2n}$$
$$=\frac{(-1)^n 2^n\lfloor n}{\lfloor 2n}t^{2n}\left\{1-\frac{t^2}{(2n+1)\cdot 2\cdot\lfloor 1}+\frac{t^4}{(2n+1)(2n+3)\cdot 2^2\cdot\lfloor 2}-\frac{t^6}{(2n+1)(2n+3)(2n+5)\cdot 2^3\cdot\lfloor 3}\right.$$
$$\left.+\cdots\right\}, \tag{9}$$

which is evidently convergent in the whole t-plane, and thus defines an integral transcendental function. Moreover $V_n(t)$ satisfies the following differential equation[5]

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[4] If $f(\xi)$ or $F(\xi)$ behaves as $1/\xi$ at $\xi=0$, so that not regular, we may put $\xi=\xi'+\beta$; now that $\frac{\partial v}{\partial\xi}=\frac{\partial v}{\partial\xi'}$, and $f(\xi')$, $F(\xi')$ become regular at $\xi'=0$, the same treatment is still possible.

[5] The equation (10) is different from ordinary equations, such as Legendre, Bessel, Gauss; and therefore it seems to deserve consideration in detail.

$$\frac{d^2y}{dt^2}-\frac{2n}{t}\frac{dy}{dt}+\left(1+\frac{2n}{t^2}\right)y=0 \tag{10}$$

and besides the following identities

$$\frac{V_n}{t^2}-\frac{V_n'}{t}=V_{n-1} \tag{11}$$

and

$$V_n''+V_n=-2nV_{n-1}. \tag{12}$$

Using these relations, we can show that the expression (7) really satisfies the equation (4). In fact, upon differentiating (7) partially with respect to t twice, we have

$$\frac{\partial^2 v}{\partial t^2}=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)V_n''(t)-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)\left[\frac{V_{n+1}''(t)}{t}-2\frac{V_{n+1}'(t)}{t^2}+2\frac{V_{n+1}(t)}{t^3}\right].$$

Adding this to (7), we get

$$\frac{\partial^2 v}{\partial t^2}+v=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(\xi)\left[V_n''(t)+V_n(t)\right]-\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)$$
$$\times\left[\frac{V_{n+1}''(t)}{t}-2\frac{V_{n+1}'(t)}{t^2}+2\frac{V_{n+1}(t)}{t^3}+\frac{V_{n+1}(t)}{t}\right],$$

in which on account of (12) and (11) the first square bracket reduces to $-2nV_{n-1}(t)$, while the second to $-\frac{2n}{t}V_n(t)$. Or, on factorizing $-2h$, it becomes

$$\frac{\partial^2 v}{\partial t^2}+v=-2h\left[\sum_{m=0}^{\infty}\frac{h^m}{m!}f^{(m+1)}(\xi)V_m(t)-\sum_{m=0}^{\infty}\frac{h^m}{m!}g^{(m+1)}(\xi)\frac{V_{m+1}(t)}{t}\right]$$
$$=-2h\frac{\partial v}{\partial \xi}, \qquad \text{Q. E. D.}$$

Next we shall show that the supplementary conditions (6) are satisfied by (7). Really, since $V_0(0)=1$ and $V_n(0)=0$, $n=1,2,3,\cdots$ as well as $\left(\frac{V_{n+1}(t)}{t}\right)_0=0$, &c, we see immediately that

$$v_{t=0}=f^{(0)}(\xi)=f(\xi),$$

and

$$\left(\frac{\partial v}{\partial t}\right)_0=\sum_{n=0}^{\infty}\frac{h^n f^{(n)}(\xi)}{n!}V_n'(0)+\sum_{n=0}^{\infty}\frac{h^n}{n!}g^{(n)}(\xi)V_n(0)=g(\xi).$$

Thus the supplementary conditions are all fulfilled.

Furthermore we can show the convergency of (7). Making use of Stirling's formula, when n is sufficiently great, we have for (9)

$$|V_n(t)| \simeq \frac{2^n \cdot n!}{(2n)!}|t|^{2n} \simeq \frac{2^n \cdot n^n \cdot \sqrt{n}}{2^{2n} \cdot n^{2n} \cdot \sqrt{2n}} e^n |t|^{2n} < \frac{1}{\sqrt{2}}\left(\frac{eR^2}{2n}\right)^n, \qquad (13)$$

where R denotes the radius of an arbitrary but fixed circle described in t-plane, origin as centre. We have assumed that $f(\xi)$ is regular within a domain D containing $\xi = 0$. Hence the inside of $\bar{D}$ can be covered by a finite number of circles, ξ being the centre of one of them, and let the radius be ρ. We have then $\max\limits_{|z-\xi|=\rho} |f(z)| \leq \max\limits_{z \text{ in } D} |f(z)| = M$, so that

$$\left|\frac{f^{(n)}(\xi)}{n!}\right| < \frac{M}{\rho^n} \leq \frac{M}{r^n}, \qquad (14)$$

where r denotes the least value among those radii of the covering circles and surely finite (>0 and not infinitely small).[6] Now we can prove the uniform convergency of the first summation in (7) as follows:

From (13) and (14) we get for $|t| < R$ and ξ in D

$$|R_{n_0}(t, \xi)| \equiv \left|\sum_{n=n_0}^{\infty} \frac{h^n}{n!} f^{(n)}(\xi) V_n(t)\right| \leq \frac{M}{\sqrt{2}} \sum_{n=n_0}^{\infty} \left(\frac{ehR^2}{2rn}\right)^n,$$

where the number in the last round bracket may be made $< \varepsilon$ by taking n sufficiently large, say $n \geq n_0$, and thus n_0 can be chosen independently of t and ξ. Consequently

$$|R_{n_0}(t, \xi)| \leq \frac{M}{\sqrt{2}} \sum_{n=n_0} \varepsilon^n = \frac{M\varepsilon^{n_0}}{\sqrt{2}\,(1-\varepsilon)} \qquad (n \geq n_0),$$

and again the last side itself can be made $<$ any prescribed small positive number ε', if we take n_0 sufficiently great, say $n_0 > n_1$. Therefore the first summation in (7) is certainly uniformly convergent. A similar argument could be made about the second summation of (7). Hence the whole expression (7) should be regular in the vicinity of $\xi = 0$, $t = 0$.

Returning to previous letters, the solution of (2) satisfying the supplementary conditions (5) is given by

$$u = e^{-b\eta} \sum_{n=0}^{\infty} \left(\frac{a}{\mu^2}\right)^n \left\{\frac{f^{(n)}(\xi)}{n!} V_n(\mu\eta) - \frac{F^{(n)}(\xi) + bf^{(n)}(\xi)}{n!} \frac{V_{n+1}(\mu\eta)}{\mu^2\eta}\right\}. \qquad (15)$$

It is convenient to take $-\mu^2 = \nu^2 = b^2 - c$, if μ^2 is real negative, and to write (15) in the form

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[6] Or more precisely we may argue as follows. Conceive an inner domain $D_1 \subset D$, which lies wholly inside D, yet almost coincides with it; thus $D_1 = \{z \mid |f(z)| < M,\ m(D - D_1) < \varepsilon\}$. Let the second inner domain be $D_2 \subset D_1$, whose boundary C_2 is apart ferm C_1, the boundary of D_1, at a distance δ (>0 however small). Now surely D_2 can be covered by a finite number of circles with the property enunciated in text.

$$u = e^{-l\eta}\left\{\sum_{n=0}^{\infty}\left(-\frac{a}{\nu^2}\right)^n \frac{f^{(n)}(\xi)}{\underline{|n}}\, W_n(\nu\eta) + \sum_{n=0}^{\infty}\left(-\frac{a}{\nu^2}\right)^n \frac{F^{(n)}(\xi)+bf^{(n)}(\xi)}{\underline{|n}}\, \frac{W_{n+1}(\nu\eta)}{\nu^2\eta}\right\}, \tag{16}$$

where

$$W_n(\tau) = V_n(i\tau) = \frac{2^n\cdot\tau^{2n}}{\underline{|2n}} \sum_{m=0}^{\infty} \frac{\underline{|m+n}}{\underline{|m}\ \underline{|2m+2n}}\, \tau^{2m}. \tag{17}$$

Lastly, if $b^2 = c$ in (3), so that $\mu = 0$, we see that

$$\lim_{\mu\to 0} \frac{V_n(\mu\eta)}{\mu^{2n}} = \frac{(-1)^n\cdot 2^n\cdot \underline{|n}}{\underline{|2n}}\, \eta^{2n},$$

by (9). Substituting this value in (15), we obtain, as the solution of (2) in the case $\mu = 0$,

$$u = e^{-l\eta}\left[\sum_{n=0}^{\infty} \frac{(-2a)^n}{\underline{|2n}} f^{(n)}(\xi)\eta^{2n} + \sum_{n=0}^{\infty} \frac{(-2a)^n}{\underline{|2n+1}} \{F^{(n)}(\xi)+bf^{(n)}(\xi)\}\, \eta^{2n+1}\right]. \tag{18}$$

Example 1. $\dfrac{\partial^2 u}{\partial\eta^2} + 2\dfrac{\partial u}{\partial\xi} + 2\dfrac{\partial u}{\partial\eta} + cu = 0$, where $c = 2$ or 1 or 0, the supplementary conditions being $u = f(\xi)$, $\dfrac{\partial u}{\partial\eta} = F(\xi)$, when $\eta = 0$.

(i) If $c = 2$, we have $\mu^2 = c - b^2 = 1$. Hence, setting $a = b = \mu^2 = 1$ in (15), we obtain the general solution

$$u = e^{-\eta}\left\{\sum_{n=0}^{\infty} \frac{f^{(n)}(\xi)}{\underline{|n}}\, V_n(\eta) - \frac{f^{(n)}(\xi)+F^{(n)}(\xi)}{\underline{|n}}\, \frac{V_{n+1}(\eta)}{\eta}\right\},$$

where

$$V_n(\eta) = \frac{(-1)^n\cdot 2^n\cdot\underline{|n}}{\underline{|2n}}\left\{\eta^{2n} - \frac{\eta^{2n+2}}{(2n+1)\cdot 2\cdot\underline{|1}} + \frac{\eta^{2n+4}}{(2n+1)(2n+3)\cdot 2^n\cdot\underline{|2}} - \cdots\right\}.$$

(ii) If $c = 1$, so $\mu^2 = c - b^2 = 0$, and we ought to take (18). Hence

$$u = e^{-\eta}\left[\sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^n}{\underline{|2n}} f^{(n)}(\xi)\eta^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^n}{\underline{|2n+1}} \{f^{(n)}(\xi)+F^{(n)}(\xi)\}\, \eta^{2n+1}\right].$$

(iii) If $c = 0$, then $\nu^2 = b^2 - c = 1$. So that we get by (16) and (17)

$$u = e^{-\eta}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n f^{(n)}(\xi)}{\underline{|n}}\, W_n(\eta) + \sum_{n=0}^{\infty} (-1)^n\, \frac{f^{(n)}(\xi)+F^{(n)}(\xi)}{\underline{|n}}\, \frac{W_{n+1}(\eta)}{\eta}\right\},$$

where

$$W_n(\eta) = \frac{2^n\cdot\underline{|n}}{\underline{|2n}}\left\{\eta^{2n} + \frac{\eta^{2n+2}}{(2n+1)\cdot 2\cdot\underline{|1}} + \frac{\eta^{2n+4}}{(2n+1)(2n+3)\cdot 2^2\cdot\underline{|2}} + \cdots\right\}.$$

Example 2. $\dfrac{\partial^2 u}{\partial\eta^2} = \dfrac{\partial u}{\partial\xi}$.

This corresponds to the case $a=-\frac{1}{2}$, $b=c=0$ in (2), and evidently bestows the differential equation of conduction of heat in a linear body with infinite length, where η denotes the distance, while ξ means time! Our solution is nothing to do for well known classical thermal differential equations, such as e. g. with the initial condition $u=\varphi(\eta)$ when $\xi=0$, and besides the boundary condition $u=0$ at $\eta=\pm\infty$. Yet it may be interpretted physically as follows: The supplementary conditions (5) express that the temperature at origin $(=u_{\eta=0})$ in any time ξ, as well as the thermal gradient there $\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}$ are confined to be $f(\xi)$ and $F(\xi)$ respectively, both being given functions of time ξ. As its solution under said conditions (the conditions at origin), we obtain from (18)

$$u=\sum_{n=0}^{\infty}\frac{1}{\lfloor 2n}f^{(n)}(\xi)\eta^{2n}+\sum_{n=0}^{\infty}\frac{1}{\lfloor 2n+1}F^{(n)}(\xi)\eta^{2n+1},$$

which gives the temperature at the distance η in any time ξ.