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NOTES ON GENERAL ANALYSIS (III): ON THE NORM OF ANALYTIC FUNCTIONS

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In classical analysis, the absolute values of functions of complex number are closely connected with their analyticity. If the absolute values of a regular function are a constant in a neighbourhood of a point, it is a constant on whole domain on which it is defined and regular. Therefore, if the absolute values of a function are a constant on whole domain, it is a constant. In complex-Banach-spaces, the norms play the part of the absolute values of complex numbers, but the norms of analytic functions do not seem to be so closely connected with their analyticity as that of the absolute values of complex numbers. That is, there exist functions whose norms are constants on their domains and yet are not constants. We shall call such a function "*an analytic function of norm-constant*". First of all, the examples of the analytic function of norm-constant are reported and then the necessary and sufficient conditions that a function should be an analytic function of norm-constant are researched, in §1. In §2, the variation of the norm of an analytic function is researched using the method of $M(r)$ in classical analysis. The variation of the extended $M(r)$ characterizes the theory of functions in complex-Banach-spaces.

§1. Analytic functions of norm-constant

Let $x_{11}, x_{12}, x_{21}, x_{22}$ be complex numbers. The set of matrixes of 2-2-types $X=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ becomes a complex-Banach-space Ω, when we define $\|X\|=$ $\mathrm{Max}(|x_{11}|, |x_{12}|, |x_{21}|, |x_{22}|)$. Let

$$Y=f(X)=\begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}+\rho\begin{pmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

where $0<\rho<\infty$, then $f(X)$ is an Ω-valued function defined on Ω. Clearly, $f(X)$ is a continuous function. Let α be a complex variable, then

$$f(X+\alpha Y)=\rho\begin{pmatrix} 1 & x_{12} \\ x_{21} & x_{22} \end{pmatrix}+\rho\begin{pmatrix} 0 & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\alpha\,.$$

This shows that $f(X)$ is G-differentiable. Therefore, $f(X)$ is analytic on whole spaces. We investigate the variety of the norm of $f(X)$ in $\|X\| \leqq 1$. When $\|X\| \leqq 1$, $|x_{ij}| \leqq 1$, where $i, j = 1, 2$. Therefore, we have

$$\begin{aligned}\|f(X)\| &= \left\| \rho \begin{pmatrix} 1 & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right\| \\ &= \rho \operatorname{Max}(1, |x_{12}|, |x_{21}|, |x_{22}|) \\ &= \rho .\end{aligned}$$

Thus we see that the norm of $f(X)$ is a constant ρ, when $\|X\| \leqq 1$, but $f(X)$ is not a constant. Moreover, $\|f(x)\| \equiv \rho$, when $|x_{11}| < \infty$, $|x_{12}| \leqq 1$, $|x_{21}| \leqq 1$, and $|x_{22}| \leqq 1$.

Now, an another example is reported. Let N be an arbitrary positive number and put $f(X) = \left(\frac{N+x_1}{2N}, \left(\frac{N+x_2}{2N}\right)^{\frac{1}{2}}, \dots\dots, \left(\frac{N+x_n}{2N}\right)^{\frac{1}{n}}, \dots\dots \right)$, where $X = (x_1, x_2, \dots\dots, x_n, \dots\dots)$ varies in the sphere $\|X\| < N$ in complex-l_2-spaces and $f(X)$ takes its values in complex-D_ω-spaces. $\left(\frac{N+x_n}{2N}\right)^{\frac{1}{n}}$ is considered on one branch of its Riemann surfaces. Then $f(X)$ is an one-valued analytic function and $\|f(X)\| = \sup_{1 \leq n} \left| \frac{N+x_n}{2N} \right|^{\frac{1}{n}} = 1$, when $\|X\| < N$ in complex-l_2-spaces.

A function defined in a bounded domain is not necessary a constant even if the norm of the function is a constant on whole domain. Now, we investigate of analytic functions whose norms are constants on whole spaces. Let E_1 and E_2 be two complex-Banach-spaces.

Theorem 1. *Let E_2-valued function $f(x)$ be an analytic function defined for all finite values in E_1. If $\|f(x)\| = 0(\|x\|^\kappa)$ as $\|x\| \to \infty$, $f(x)$ is a polynomial of degree k at most.*

Proof. Since $f(x)$ is analytic for all finite values in E_1, $f(\alpha x)$ is expressed as follows

$$f(\alpha x) = \sum_{n=0}^{\infty} h_n(x) \alpha^n,$$

for an arbitrary point x and an arbitrary complex number α, where $h_n(x)$ is a homogeneous polynomial of degree n. Let C be a circle $|\alpha| = R$ on the complex-α-plane, then we have

$$\begin{aligned}h_n(x) &= \frac{1}{2\pi i} \int_c \frac{f(\alpha x)}{\alpha^{n+1}} d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta} x)}{R^n e^{in\theta}} d\theta ,\end{aligned}$$

where $\alpha = Re^{i\theta} (0 \leqq \theta \leqq 2\pi)$. Therefore, we have

$$||h_n(x)|| \leqq \frac{1}{2\pi} \int_0^{2\pi} \frac{||f(Re^{i\theta}x)||}{R^n} d\theta$$
$$= 0(R^k ||x||^k)/R^n,$$

for sufficiently larg R, because $||f(x)|| = 0(||x||^k)$, as $||x|| \to \infty$. R can be taken as large as we like, $||h_n(x)|| = 0$, when $n > k$. Since x is an arbitrary point of E, $h_n(x) \equiv 0$, when $n > k$. Thus we have $f(x) = \sum_{n=0}^{k} h_n(x)$, which is a polynomial of degree k at most.

Corollary[*)] **1.** *If $f(x)$ is analytic and satisfies $||f(x)|| \leqq M$ on whole spaces, then $f(x)$ is a constant.*

Proof. Appealing to Theorem 1 for $k = 0$, we have $f(x) \equiv f(0)$.

Corollary 2. *If $f(x)$ is analytic and $||f(x)|| \equiv c$ on whole spaces, then $f(x)$ is a constant.*

An analytic function whose norm is a constant on whole space is a constant but a function is not necessarily a constant, even if the norm of the function is a constant on a bounded domain. It seems to be a character of the function as well as the boundedness of the function that the norm of the function is a constant. Thus, the necessary and sufficient condition that the norm of the function should be a constant must be investigated. An open and connected set is called a domain.

Lemma. *If $f(x)$ is analytic on a domain D, the set of x which satisfies $||f(x)|| = c$ is a (relative) closed set in D, where c is a constant.*

Proof. Suppose that a sequence $\{x_n\}$ converges to x_0 in D and satisfies $||f(x_n)|| = c$ for $n = 1, 2, 3, \ldots\ldots$. Since $f(x)$ is continuous and satisfies

$$\Big| ||f(x_0)|| - ||f(x_n)|| \Big| \leqq ||f(x_n) - f(x_0)||, \quad \lim_{n\to\infty} ||f(x_n)|| = ||f(x_0)||$$

and we have $||f(x_0)|| = c$. This completes the proof.

Theorem 2. *If an analytic function $f(x)$ defined on D satisfies $||f(x)|| \leqq M$ and moreover $||f(x_0)|| = M$ for a point x_0 in D, $||f(x)|| \equiv M$. (The inverse is also true.)*

Proof. Let S be a set of point x which satisfies $||f(x)|| = M$ in D. S is not a null set by the assumption. Let x_0 be an arbitrary point of S. Since x_0 is an inner point of D, there exists a neighbourhood $V(x_0)$, which is a set of point x satisfying $||x - x_0|| < \rho$, in D. Suppose that $S \not\supset V(x_0)$, then there exists a point y in $V(x_0)\frown CS$. Since $f(x)$ is analytic in D,

$$f(x_0)=\frac{1}{2\pi i}\int_c \frac{f(x_0+\alpha(y-x_0))}{\alpha}\,d\alpha\,,$$

where C is a circle $|\alpha|=1$. Clearly $x+\alpha(y-x_0)\in V(x_0)$ when $\alpha\in C$. Let $\alpha=e^{i\theta}(0\leqq\theta\leqq 2\pi)$, then we have

$$||f(x_0)||\leqq\frac{1}{2\pi}\int_0^{2\pi}||f(x_0+e^{i\theta}(y-x_0))||\,d\theta.$$

Since $f(x)$ is continuous at y and satisfies $||f(y)||<M$, there exist a positive number ε and δ satisfying $||f(x_0+e^{i\theta}(y-x_0))||<M-\varepsilon$ when $|\theta|\leqq\delta$.
Then we have,

$$\begin{aligned}||f(x_0)||&\leqq\frac{1}{2\pi}\int_{-\delta}^{+\delta}||f(x_0+e^{i\theta}(y-x_0))||d\theta+\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}||f(x_0+e^{i\theta}(y-x_0))||d\theta\\&\leqq M-\frac{\varepsilon\delta}{\pi}\end{aligned}$$

contradicting to that $x_0\in S$. Therefore, $S\supset V(x_0)$ and we see that S is an open set. If $D\neq S$, $D=S+D\frown CS$, where $D\frown CS$ is an open set, because S is a closed set by Lemma. This contradicts to that D is connected. Therefore, $D=S$, which shows that the norm of $f(x)$ is a constant M on D.

Theorem 3. (*The extended theorem of Lindelöf.*) *If an analytic function $f(x)$ defined on a bounded domain D satisfies the followig conditions*: (1) *there exists a neighbourhood $V(x)$ of x for an arbitrary positive number ε and an arbitrary point x on the boundary of D and $||f(y)||<M+\varepsilon$ when $y\in V(x)\frown D$,* (2) *$||f(x_0)||=M$ for a point x_0 in D, then $||f(x)||\equiv M$ on D. Therefore, if $||f(x)||$ is not a constant, $||f(x)||<M$ in D.*

Proof. Let y be a point in D and α be a complex variable. $f(\alpha y)$ is an analytic function of α while $\alpha y\in D$. Plainly, a set E composed of α which satisfies $\alpha y\in D$ is an open set. Now, let S be a component of 1 in E, then S is a bounded domain in α-plane. Let a boundary of S be Γ, then, if $\alpha_0\in\Gamma$, $\alpha_0 y$ is a boundary point of D. From the assumption (1), for an arbitrary positive number ε, there exists a neighbourhood $V(\alpha_0 y)$ and $||f(x)||<M+\varepsilon$ when $x\in V(\alpha_0 y)\frown D$. For a sufficiently small positive number δ which is smaller than the distance between 1 and Γ, $\alpha y\in V(\alpha_0 y)$, when $|\alpha-\alpha_0|\leqq\delta$ and then Γ is covered by such circles. Since Γ is a closed bounded set on α-plane, it is covered by such circles of finite numbers. Accordingly, there are some domain S_i surrounded by the arcs of finite numbers. Among S_i,

let S_0 be a domain which includes 1. Since $||f(\alpha y)||$ is subharmonic about α, $||f(\alpha y)||$ takes its maximum on the boundary of S_0 and we see that $||f(\alpha y)|| < M+\varepsilon$, when $\alpha \in S_0$. Since $S_0 \ni 1$, $||f(y)|| < M+\varepsilon$ and then $||f(y)|| \leqq M$, because ε is an arbitrary positive number. Thus we see that $||f(x)|| \leqq M$ on the whole domain D, because y is an arbitrary point of D. From the assumption (2) and Theorem 2, $||f(x)|| \equiv M$ on D. Finally, if $||f(x)||$ is not a constant, the assumption (2) must be false, that is, $||f(x)|| < M$ for all x in D. This completes the proof.

It is plain that Theorem 2 and Theorem 3 are the necessary conditions $||f(x)||$ to be a constant.

§ 2. Extended $M(r)$

Let D be a domain in E_1 being expected to include the origin without losing generality and E_2-valued function $f(x)$ be analytic in D.

Definition 1. *A positive number σ is called a radius of norm-constant of $f(x)$ (with respect to 0), if it satisfies following conditions*: (1) $||f(x)|| \equiv C$ *for all x in $||x|| \leqq \sigma$, where C is a constant*, (2) *for an arbitrary positive number ε, there exists in the sphere $||x|| < \sigma+\varepsilon$ at least a point x on which $||f(x)|| \neq C$.*

The sphere of radius σ is called *the sphere of norm-constant of $f(x)$*.

Definition 2. *When x is an arbitrary point on $||x|| = 1$, $r(x)$ is defined as the upper limit of r such that $||f(\alpha x)|| = C$ (which is a constant) when $|\alpha| \leqq r$.*

Then we have the following theorem.

Theorem 4. *Put $\operatorname{Inf}_{||x||=1} r(x) = \sigma$, then σ is the radius of norm-constant of $f(x)$.*

Proof. We prove this theorem for $0 < \sigma < \infty$, because, when $\tau = 0$ or ∞, we can discuss as well. Since $||f(0)|| = C$, C is independent of the variation of x on $||x|| = 1$. When $|\alpha| \leqq r(x)$, $||f(\alpha x)|| = C$, since $||f(\alpha x)||$ is a continuous function of α. Let y be an arbitrary point in $0 < ||x|| \leqslant \sigma$. Put $x = \dfrac{y}{||y||}$, then $||x|| = 1$ and we have $||f(||y||x)|| = C$, since $||y|| \leqq \sigma = \operatorname{Inf}_{||x||=1} r(x) \leqq r(x)$ and $||f(\alpha x)|| = C$ for $|\alpha| \leqq r(x)$. On the other hand, since $y = ||y||x$, $||f(y)|| = C$. Since y is an arbitrary point in $0 < ||x|| \leqslant \sigma$, $||f(x)|| = C$ for all values on $0 < ||x|| \leqq \sigma$. Considering the fact that $||f(0)|| = C$, $||f(x)|| \equiv C$ for $||x|| \leqq \sigma$. From the definition of σ, we see that there exists at least a point x_0, which satisfies $||f(x_0)|| \neq C$, in $||x|| < \sigma+\varepsilon$ for an arbitrary positive number ε. Therefore, σ is a radius of norm-constant of $f(x)$.

Theorem 5. *Let $f(x)$ be analytic in a domain D and $f(0) = 0$. If $\sigma > 0$,* *then* $f(x)$ *is a constant* 0, *where* σ *is the radius of norm-constant of* $f(x)$.

Proof. From the definition of the radius of norm-constant, $||f(x)||=C$, when x lies in $||x||\leqq\sigma$, where C is a constant. Then $C=||f(0)||=0$. Therefore, $f(x)\equiv 0$ on $||x||\leqq\sigma$. Let the largest sphere including 0 in D be $||x||<R$. If $\sigma<R$ and y is an arbitrary point in $||x||<R$, $f(y)=\sum_{n=1}^{\infty}h_n(y)$. Then we have

$$h_n(y)=\frac{1}{2\pi i}\int_c \frac{f(\alpha y)}{\alpha^{n+1}}d\alpha,$$

where C is a circle such that $|\alpha|=\dfrac{\sigma}{||y||}$. While, $f(\alpha y)=0$ when $||\alpha y||=\sigma$. Then we have $h_n(y)=0$, where $n=1, 2, 3, \ldots\ldots$, and we have $f(y)=0$. This shows that $f(x)\equiv 0$ in $||x||<R$, since y is an arbitrary point in $||x||<R$. By the analytic continuation, $f(x)\equiv 0$ on D.

Corollary. *Let* $f(x)$ *be analytic in a domain* D *and put* $g(x)=f(x)-f(0)$. *If a radius of norm-constant of* $g(x)$ *is not zero, then* $f(x)$ *is a constant on* D.

Proof. Appealing to Theorem 5, $g(x)\equiv 0$ on D and then $f(x)\equiv f(0)$ on D.

Definition 3. *For an arbitrary point* x *on* $||x||=1$, *we define* $M(r, x)=\operatorname{Max}_{|\alpha|=r}||f(\alpha x)||$, *where* α *is a complex variable. Put* $M(r)=\operatorname{Sup}_{||x||=r}||f(x)||$.

Theorem 6. *If* $f(x)$ *is analytic in* $||x||<R$ *and* $r<R$, (1) $M(r)=\operatorname{Sup}_{||x||\leq r}||f(x)||$, (2) $M(r)=\operatorname{Sup}_{||x||=1}M(r, x)$, (3) $M(r)\leqq M(r')$ *when* $r\leqq r'$, *where* $r'<R$.

Proof of (1). Clearly $M(r)\leq\operatorname{Sup}_{||x||\leq r}||f(x)||$. If $M(r)<\operatorname{Sup}_{||x||\leq r}||f(x)||$, there exists in $||x||<r$ a point x_0 which satisfies $M(r)<||f(x_0)||$. $f(\alpha x_0)$ is an analytic function of α in $|\alpha|<\dfrac{R}{||x_0||}$ and then $||f(\alpha x_0)||$ is subharmonic in $|\alpha|\leqslant\dfrac{r}{||x_0||}(>1)$. Therefore, $||f(\alpha x_0)||$ takes its maximum on the boundary $|\alpha|=\dfrac{r}{||x_0||}$ contradicting to the fact that $||f(x_0)||>M(r)\geqslant||f(\alpha x_0)||$, where $|\alpha|=\dfrac{r}{||x_0||}$. Thus we see that $M(r)=\operatorname{Sup}_{||x||\leq r}||f(x)||$.

Proof of (2). Since $||f(\alpha x)||$ is a continuous function of α, there exists α_0 which satisfies $||f(\alpha_0 x)||=\operatorname{Max}_{|\alpha|=r}||f(\alpha x)||=M(r, x)$, $|\alpha_0|=r$. Then $M(r, x)=||f(\alpha_0 x)||\leqq M(r)$, for an arbitrary point x on $||x||=1$. Hence, $\operatorname{Sup}_{||x||=1}M(r, x)\leqq M(r)$. On the other hand, $||f(y)||\leqq M(r, x)$, when $||y||=r$ and $y=rx$. Hence, $||f(y)||\leqq\operatorname{Sup}_{||x||=1}M(r, x)$ for an arbitrary point y on $||x||=r$. Then we have $M(r)\leqq\operatorname{Sup}_{||x||=1}M(r, x)$. Thus we see that $M(r)=\operatorname{Sup}_{||x||=1}M(r, x)$.

Proof of (3). For an arbitrary positive number ε, there exists a point x_0 on $||x||=r$ satisfying $M(r)-\varepsilon\leqq||f(x_0)||$. Appealing to (1), $||f(x_0)||\leqq M(r')$, since $||x_0||=r<r'$. Then we have $M(r)-\varepsilon<M(r')$ for an arbitrary positive number ε. Hence, $M(r)\leqq M(r')$.

Theorem 7. *Let σ be the upper limit of r which satisfies $M(r)=||f(0)||$. Then σ is the radius of norm-constant of $f(x)$, where $f(0)\neq 0$.*

Proof. For an arbitrary positive number ε, there exists r which satisfies $\sigma-\varepsilon<r$ and $M(r)=||f(0)||$. By Theorem 6-(1), $||f(x)||\leqq M(r)$, when $||x||\leqq r$. Thus we have $||f(x)||\equiv||f(0)||$ on $||x||\leqq r$, appealing to Theoren 2. Let x_0 be an arbitrary point on $||x||=\sigma$ and x' be a point being included in both spheres $||x-x_0||\leqq\varepsilon$ and $||x||\leqq r$, then $||f(x')||=||f(0)||$. Since $||f(x)||$ is continuous, $||f(0)||=\lim_{x'\to 0}||f(x')||=||f(x_0)||$. Thus we see that $||f(x)||\equiv||f(0)||$, when x lies on $||x||\leqq\sigma$. From the definition of σ, $||f(x)||\not\equiv||f(0)||$ on $||x||\leqq r$, when $r>\sigma$. This completes the proof.

Theorem 8. (*Extended Hadamard's three spheres theorem*). *If $r_1<r_2<r_3$,*

$$M(r_2)\leqq M(r_1)^{\frac{\log r_3-\log r_2}{\log r_3-\log r_1}}M(r_3)^{\frac{\log r_2-\log r_1}{\log r_3-\log r_1}}.$$

Proof. Let x be an arbitrary point on $||x||=1$ and ρ be a positive number which satisfies $r_1^\rho M(r_1, x)=r_3^\rho M(r_3, x)$. Put $F(\alpha)=\alpha^\rho f(\alpha x)$, where α is a complex variable. Since $||F(\alpha)||$ is a subharmonic function of α, we have

$$M(r_2, x)\leqq M(r_1, x)^\theta M(r_3, x)^{1-\theta},$$

as well as the case of complex functions, where $\theta=\dfrac{\log r_3-\log r_2}{\log r_3-\log r_1}$ and $1-\theta=\dfrac{\log r_2-\log r_1}{\log r_3-\log r_1}$. By appealing to Theorem 6-(2), we have

$$\begin{aligned}M(r_2)&=\sup_{||x||=1}M(r_2, x)\leqq\sup_{||x||=1}(M(r_1, x)^\theta M(r_3, x)^{1-\theta})\\&\leqq\sup_{||x||=1}M(r_1, x)^\theta\sup_{||x||=1}M(r_3, x)^{1-\theta}\\&=(\sup_{||x||=1}M(r_1, x))^\theta(\sup_{||x||=1}M(r_3, x))^{1-\theta}\\&=M(r_1)^\theta M(r_3)^{1-\theta},\end{aligned}$$

since t^θ and $t^{1-\theta}$ is continuous when $t>0$. This completes the proof.

Theorem 9. *If $r_1<r_2$ and $M(r_1)=M(r_2)$, $f(x)$ is the function of norm-constant on $||x||\leqq r_2$. Therefore, if $f(x)$ is not a function of norm-constant in $||x||\leqq r_2$, $M(r_1)<M(r_2)$, when $r_1<r_2$.*

Proof. If $0<r<r_1$, $M(r_1)\leqq M(r)^\theta M(r_2)^{1-\theta}$ by Theorem 8, where $\theta=\dfrac{\log r_2-\log r_1}{\log r_2-\log r}$ and $1-\theta=\dfrac{\log r_1-\log r}{\log r_2-\log r}$. Since $M(r_1)=M(r_2)$, which is not

less than $||f(0)||$ by Theorem 6-(1), $M(r_1) \leqq M(r)^\theta M(r_1)^{1-\theta}$, then $M(r_1)^\theta \leqq M(r)^\theta$. Since $0<\theta<1$, $M(r_1) \leqq M(r)$. On the other hand, $M(r) \leqq M(r_1)$, by Theorem 6-(3). Then we have $M(r)=M(r_1)=M(r_2)$. Since $f(x)$ is continuous, for an arbitrary positive number ε, there exists a positive number δ such that $||f(x)-f(0)||<\varepsilon$, when $||x||<\delta$. If $r<\delta$, $M(r)=\operatorname{Sup}_{||x||=r} ||f(x)|| \leqq ||f(0)||+\varepsilon$. Then we have, $M(r_2) \leqq ||f(0)||+\varepsilon$. Since ε is an arbitrary positive number, $M(r_2)=||f(0)||$. Appealing to Theorem 2, $f(x)$ is the function of norm-constant on $||x|| \leqq r_2$. Therefore, if $f(x)$ is not a function of norm-constant on $||x|| \leqq r_2$, $M(r_1)<M(r_2)$.

Corollary. *If $f(x)$ is analytic in $||x||<R$ and the norm of $f(x)$ is a constant, when $0<r<||x||<R$, then $f(x)$ is a function of norm-constant on $||x||<R$.*

Proof. Let $r<r_1<r_2<R$, $M(r_1)=M(r_2)$. Appealing to Theorem 9, $f(x)$ is a function of norm-constant on $||x|| \leqslant r_2$. Since r_2 can be taken as close as we like to R, $f(x)$ is a function of norm-constant on $||x|| \angle R$.

Definition 4[1]. *If a positive number λ satisfies the following conditions* (1) *if $0<r<\lambda$, $f(x)$ is analytic and bounded in the sphere defined by $||x||<r$,* (2) *if $r>\lambda$, $f(x)$ can not be analytic and bounded in the sphere defined by $||x||<r$, then λ is called a radius of bound of $f(x)$.*

Theorem 10. *Let λ be the upper limit of r which satisfies $M(r)<\infty$, then λ is the radius of bound of $f(x)$.*

Proof. For an arbitrary positive number r, which satisfies $r<\lambda$, $M(r)<\infty$. By Theorem 6-(1), $||f(x)|| \leqq M(r)$, when $||x|| \leqq r$. That is, $f(x)$ is bounded on $||x||<r$. By the definition of λ, it is clear that there does not exist M which satisfies $||f(x)|| \leqq M$ when $||x||<r$, where $r>\lambda$. Thus we see that λ is the radius of bound of $f(x)$.

Theorem 11. *If the radius of norm-constant is infinite, $f(x)$ is a constant.*

Proof. Since $||f(x)||$ is a constant on whole space, $f(x)$ is a constant by Corollary 2 of Theorem 1.

Thus we see that $M(r)$ varies generally as follows

$M(r)=||f(0)||$, when $0 \leqq r \leqq \sigma$,

$=$ steadily increases with r in the stricter sense, when $\sigma<r<\lambda$ and tends to $+\infty$, as r tends to λ,

$=+\infty$, when $\lambda \leqq r<\tau$, where τ is the radius of analyticity[1)-2)] of $f(x)$.

The following example will practically show the above stated variety of $M(r)$. Let Ω and Ω' be complex-l_2-spaces and complex-D_ω-spaces respectively. $X=(x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\ldots)$ is a point of Ω, where x_i is a complex variable $(i=1, 2, 3, \ldots\ldots)$. Put Ω'-valued functions $h_n(X)$ defined on Ω as follows

$$\begin{aligned} h_0(X) &= \left(\frac{1}{2},\ 0,\ 0,\ \ldots\ldots\ldots\ldots\right) \\ h_1(X) &= (0,\ x_1,\ x_2,\ x_3,\ \ldots\ldots) \\ h_2(X) &= (0,\ x_2^2,\ x_3^2,\ x_4^2,\ \ldots\ldots) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_n(X) &= (0,\ x_n^n,\ x_{n+1}^n,\ x_{n+2}^n,\ \ldots\ldots) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ , \end{aligned}$$

then $f(X)=\sum_{n=0}^{\infty} h_n(X)$ is analytic on whole space and has $\sigma=\frac{1}{2}$, $\lambda=1$ and $\tau=+\infty$.

First of all, we must show that $h_n(X)$ is a homogeneous polynomial of degree n. Since $h_n(X)\in\Omega'$, $\|h_n(X)\|=\operatorname{Sup}_{0\leq i<\infty}|x_{n+i}|^n$, for $n=1, 2, 3, \ldots\ldots$, where $\|X\|=\sqrt{\sum_{n=1}^{\infty}|x_n|^2}$, since $X\in\Omega$. It is clear that $h_n(X)$ is a continuous function by the definition of the norm. Since

$$h_n(X+\alpha Y)=(0,\ x_n^n,\ x_{n+1}^n,\ \ldots\ldots)+n\alpha(0,\ x_n^{n-1}y_n,\ x_{n+1}^{n-1}y_{n+1},\ \ldots\ldots)+\ldots\ldots,$$

$h_n(X+\alpha Y)$ is an analytic function of α.

$$\begin{aligned} h_n(\alpha X) &= (0,\ (\alpha x_n)^n,\ (\alpha x_{n+1})^n,\ \ldots\ldots) \\ &= \alpha^n(0,\ x_n^n,\ x_{n+1}^n,\ \ldots\ldots\ldots) \\ &= \alpha^n h_n(X). \end{aligned}$$

This shows that $h_n(X)$ is a homogeneous polynomial of degree n.

(1) Proof of $\tau=+\infty$.

$$\begin{aligned} \tau &= 1/\operatorname{Sup}_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(X)\|}^{\,1)-2)} \\ &= 1/\operatorname{Sup}_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\operatorname{Sup}_{0\leq i}|x_{n+i}|^n} \\ &= 1/\operatorname{Sup}_{\|x\|=1}\overline{\lim_{n\to\infty}}\operatorname{Sup}_{0\leq i}|x_{n+i}| \\ &= +\infty,\ \text{since } |x_n|\to 0, \text{ when } n\to+\infty. \end{aligned}$$

(2) Proof of $\lambda=1$.

$$\lambda=1/\overline{\lim_{n\to\infty}}\sqrt[n]{\operatorname{Sup}_{\|x\|=1}\|h_n(X)\|}^{\,1)}=1/\overline{\lim_{n\to\infty}}\sqrt[n]{\operatorname{Sup}_{\|x\|=1,\,0\leq i}|x_{n+i}|^n}=1.$$

(3) Proof of $\sigma=\frac{1}{2}$.

$\|f(x)\| = \operatorname{Sup}\left(\frac{1}{2},\ |\sum_{n=1}^{\infty}x_{n+i}^n|,\ i\geqq 0\right)$. When $\|X\|\leqq\frac{1}{2}$, $\sum_{n=1}^{\infty}|x_n|^2\leqq\frac{1}{4}$, then $|x_n|\leqq\frac{1}{2}$ for $n\geqq 1$. Thus we have $\sum_{n=2}^{\infty}|x_{n+i}|^n\leqq\sum_{n=2}^{\infty}|x_{n+i}|^2$. On the other hand, $\sum_{n=1}^{\infty}|x_{n+i}|^2 \leqq \frac{1}{4}$,

then $$|x_{1+i}| \leqq \sqrt{\frac{1}{4}-\sum_{n=2}^{\infty}|x_{n+i}|^2}.$$

Then $$|\sum_{n=1}^{\infty} x_{n+i}^n| \leqq |x_{1+i}| + \sum_{n=2}^{\infty}|x_{n+i}|^n \leqq \sqrt{\frac{1}{4}-\sum_{n=2}^{\infty}|x_{n+i}|^2}+\sum_{n=2}^{\infty}|x_{n+i}|^n \leqq \frac{1}{2}.^{3)}$$

Thus we have, $||f(X)|| = \frac{1}{2}$, when $||X|| \leqq \frac{1}{2}$. For an arbitrary positive number ε, put $X=\left(\frac{1}{2}+\varepsilon, 0, 0, \ldots\ldots\right)$, then $||f(X)||=\mathrm{Sup}\left(\frac{1}{2}, \frac{1}{2}+\varepsilon\right)=\frac{1}{2}+\varepsilon$. This shows that $\sigma=\frac{1}{2}$.

References.

1) I. Shimoda; On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

2) E. Hille; Functional Analysis and Semi-groups.

3) Put $\sum_{n=2}^{\infty}|x_{n+i}|^2=x$, $\sum_{n=2}^{\infty}|x_{n+i}|^n=y$, then $\sqrt{\frac{1}{4}-x}+y \leqq \frac{1}{2}$ may be proved easily for $0 \leqq y \leqq x \leqq \frac{1}{4}$.

*) See A. E. Taylor: Analytic function in general Analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. Vol. VI (1937), page 15, where Liouville's theorem was proved.