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Errata

Page 3, line 10, read "$\overset{\lambda}{R}_p$" instead of "R_p."

Page 3, line 10, read "$\overset{\nu}{L}_q$" instead of "L_q."

Page 3, line 9 from bottom, read "$\overset{\nu}{L}_p$" instead of "L_p."

Page 3, line 2 from bottom, read "$\overset{\nu}{L}_q$" instead of "L_q."

Page 4, line 1, read "$\overset{\nu}{R}_p\overset{\nu}{L}_q$" instead of "$R_pL_b$."

Page 4, line 1, read "$\overset{\nu}{L}_q\overset{\nu}{R}_p$" instead of "$\overset{\nu}{L}_q\overset{\nu}{R}_q$."

Page 4, line 3 from bottom, mark with a period in the end.

Page 5, line 7, read "$\overset{\mu}{R}_s\overset{\mu}{R}_a=\overset{\mu}{R}_p$" instead of "$\overset{\mu}{R}_s\overset{\mu}{R}_p=\overset{\mu}{R}_a$."

Page 11, line 13, read "$\lambda^{\underline{e}}$" instead of "$\lambda^{\underline{e}}$,"
"$x\ \lambda^{\underline{e}}\ \ y$" instead of "$x\ \lambda\ {}^{\underline{e}}y$."

Page 13, line 4 from bottom, read "satisfy" instead of "satisfies."

S. 17, Z. 13 v. o. statt „e^c" lies „$e^{c\eta}$."

S. 17, Z. 4 v. u. statt „$x=\alpha-1$ (<0)" lies „$x=\alpha-1$ (>0)."

S. 18, Z. 4 v. o. statt „$\alpha<1$" lies „$\alpha>1$."

S. 18, Z. 3 v. u. statt „$e^{c(\pi+\ )}$" lies „$e^{c(\pi+\tau)}$."

Page 22, line 6 from bottom, read "e_{i_1}" insted of "e_{i1}."

Page 22, line 6 from bottom, read "$e_{i_{k-1}}$" insteed of "e_{ik-1}."

Page 22, line 5 from bottom, read "e_{i_k}" instead of "e_{ik}."

Page 25, line 16, read "$(a, \vec{r})\subset M^c$" instead of "$(a, r)\vec{\subset} M^c$."

Page 26, line 12, read "$0<\mu<\lambda\leqq 1$)" instead of "$0<\mu<\lambda\leqq 1$."

Page 29, line 2, read

read "$\frac{y_0^2}{S_0}x^1+\frac{x_0^2}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2=0$" instead of "$\frac{y_0^2}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2=0$."

Page 29, line 5,

read "$\begin{pmatrix} \frac{x_0^2}{S_0} & \frac{y_0^2}{S_0} & \frac{x_0^1}{S_0} & -\frac{y_0^1}{S_0} \\ -\frac{y_0^2}{S_0} & \frac{x_0^2}{S_0} & \frac{y_0^1}{S_0} & \frac{x_0^1}{S_0} \\ -\frac{x_0^1}{S_0} & -\frac{y_0^1}{S_0} & \frac{x_0^2}{S_0} & -\frac{y_0^2}{S_0} \\ \frac{y_0^1}{S_0} & -\frac{x_0^1}{S_0} & \frac{y_0^2}{S_0} & \frac{x_0^2}{S_0} \end{pmatrix}$." instead of "$\begin{pmatrix} \frac{x_0^2}{S_0} & \frac{y_0^2}{S_0} & \frac{y_0^1}{S_0} & -\frac{y_0^1}{S_0} \\ -\frac{y_0^2}{S_0} & \frac{x_0^2}{S_0} & \frac{y_0^1}{S_0} & \frac{x_0^1}{S_0} \\ -\frac{x_0^1}{S_0} & -\frac{x_0^2}{S_0} & \frac{x_0^2}{S_0} & -\frac{y_0^2}{S_0} \\ \frac{y_0^1}{S_0} & -\frac{x_0^1}{S_0} & \frac{y_0}{S_0} & \frac{x_0^2}{S_0} \end{pmatrix}$"

Page 29, lines 9 & 10,

read "$X_2=\frac{x_0^1}{S_0}x^1+\frac{y_0^1}{S_0}y^1+\frac{x_0^2}{S_0}x_2+\frac{x_0^2}{S_0}y^2,$ $Y^2=-\frac{y_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^2}{S_0}x^2+\frac{x_0^2}{S_0}y^2$" instead of "$X^2=\frac{x_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1+\frac{x_0^1}{S_0}x^2+\frac{y_0^2}{S_0}y^2,$" $Y^2=-\frac{y_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2$.

Page 29, line 12,

read "$+i\left(-\frac{y_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^2}{S_0}x^2+\frac{x_0^2}{S_0}y^2\right)$" instea of "$+i\left(-\frac{y^{01}}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2\right)$."

Page 30, line 21, read "$e^{i\varphi}$" instead of "e^i."

Page 30, line 23, read "〔P′Q〕" instead of "〔Q′Q〕".

Page 31, line 17, read "$\begin{vmatrix} y^1 & x^2 \\ y'^1 & x'^2 \end{vmatrix}$" instead of "$\begin{vmatrix} y^1 & x^1 \\ y'^1 & x'^2 \end{vmatrix}$".

Page 33, line 19, read "$\bar{u}$, $\bar{v}$" instead of "^-u, ^-v" rspectively.

Page 35, line 2 from bottom, read "Proof" insted of "Proo‡".

Some Remarks on Semi-groups and All Types of Semi-groups of Order 2, 3.

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In this paper we shall relate first to a certain necessary and sufficient condition for an algebra to be a semi-group and some properties of its principal ideals〔1〕[0] as the addendum to our results in this Journal, vol 2, secondly to some semi-group extensions, i.e., semi-groups which contain one or two given semi-groups, and finally we shall utilize them and determine all types of semi-groups of order 2 and 3. We note that no assumption of finiteness is necessary in § 1, 2.

§ 1. The Condition and Properties of Semi-group.

It was proved in 〔2〕 that the algebra S was a semi-group if and only if it was isomorphic (anti-isomorphic) on the right (left) faithful realization system. Here we try to establish another condition which is simpler. The present notations are somewhat different from those used previously 〔3〕. The signs $\overset{\lambda}{R}_a$, $\overset{\lambda}{L}_a$ stand for two different meanings as the case may be: one is the realization of a, i.e., the mapping[1] of S into itself, $\overset{\lambda}{R}_a(x)=x\lambda a$, $\overset{\lambda}{L}_a(x)=a\lambda x$, the other is the subset, called principal ideal. The equality as the set is written $\overset{\lambda}{R}_a \underset{i}{=} \overset{\lambda}{R}_b$ to distinguish it from the equality $\overset{\lambda}{R}_a=\overset{\lambda}{R}_b$ as the mapping. While the discussion is proceeded under an operation, the sign "λ" may be omitted.

Theorem 1. *The algebra $S(\lambda)$ is a semi-group if and only if $\overset{\lambda}{R}_a\overset{\lambda}{L}_b=\overset{\lambda}{L}_b\overset{\lambda}{R}_a$ [2] for every $a, b \in S$.*

More generally,

Theorem 2. *Let λ and μ be semi-group operations defined in S. It holds that $\lambda \gtrsim \mu$ 〔4〕 if and only if $\overset{\mu}{R}_a\overset{\lambda}{L}_b=\overset{\lambda}{L}_b\overset{\mu}{R}_a$ for every $a, b \in S$.*

Proof. The theorems are easily obtained from the following.

$$\left\{\overset{\mu}{R}_a\overset{\lambda}{L}_b\right\}(x)=\overset{\lambda}{L}_b\left\{\overset{\mu}{R}_a(x)\right\}=\overset{\lambda}{L}_b(x\mu a)=b\lambda(x\mu a),$$

$$\left\{\overset{\lambda}{L}_b\overset{\mu}{R}_a\right\}(x)=\overset{\mu}{R}_a\left\{\overset{\lambda}{L}_b(x)\right\}=\overset{\mu}{R}_a(b\lambda x)=(b\lambda x)\mu a$$

---

0) Numbers in brackets 〔 〕 refer to the references at the end of the paper.

1) We called it a transformation previously 〔2〕.

2) We defined $\left\{\overset{\lambda}{R}_a\overset{\lambda}{L}_b\right\}(x)=\overset{\lambda}{L}_b\left\{\overset{\lambda}{R}_a(x)\right\}$ in 〔2〕

for every $a, b, x \varepsilon S$. It becomes Theorem 1 in case that $\lambda=\mu$.

Theorem 3. *If $x \in R_a$, then $R_x \subset R_a$, and if $x \varepsilon L_a$ then $L_x \subset L_a$.*

Proof, $x=ya$ for some $y \varepsilon S$. $zx=z(ya)=(zy)a$ for any $z \in S$; hence $R_x \subset R_a$. Similarly $L_x \subset L_a$.

Let S be a semi-group with one at least idempotent hereafter.

Theorem 4. *Let a be an idempotent of S.*

(1) *If $x \varepsilon R_a$, then $xa=x$.*

(2) *If and only if $R_a \underset{i}{=} S$, a is a right unit.*

(3) *If $R_a \underset{i}{=} \{a\}$ and $ab=b$, then $R_b \underset{i}{=} \{b\}$.*

(4) *If $R_a \underset{i}{=} S$ and $ac=ab$, then $xc=xb$ for all $x \varepsilon S$.*

In the dual cases they are similar except slight modification.

Proof. (1) Since $x=ya$ for some $y \varepsilon S$, $xa=(ya)a=y(aa)=ya=x$. (2) is evident by (1). (3) $xb=x(ab)=(xa)b=ab=b$ for all x. (4) Using (2), $xc=(xa)c=x(ac)=x(ab)=(xa)b=xb$.

§ 2 Semi-group Extensions

Let A and B be disjoint semi-groups with the operations λ and μ respectively. We shall construct some sorts of semi-proups which include A and B as the sub-semi-groups keeping the operations invariant.

The set of all pairs (x, y) where $x \varepsilon A$ and $y \varepsilon B$ is called the direct product of A and B. Its operation ν is defined as $(x, y)\, \nu\, (x', y')=(x \lambda x', y \mu y')$. Then we have without difficulty

Theorem 5. *The direct product $D(\nu)$ of semi-groups $A(\lambda)$ and $B(\mu)$ is a semi-group.*

The union C of $A(\lambda)$ and $B(\mu)$ will become a semi-group, if we give such operations as seen in the below theorems, which are all proved by dint of Theorem 1. In the following theorems we don't mention that $A(\lambda)$ and $B(\mu)$ are semi-groups, $A(\lambda) \cap B(\mu)=0$ and $C(\nu)=A(\lambda) \cup B(\mu)$.

Theorem 6. *If ν is given as:*

$$x \nu y=x \lambda y \qquad \text{for } x, y \in A$$
$$x \nu y=x \mu y \qquad \text{for } x, y \in B$$
$$x \nu y=y \nu x=y \qquad \text{for } x \in A,\ y \in B,$$

then $C(\nu)$ is a semi-group.

Before the proof we explain the notaions. By $\overset{\nu}{R_x}=\left(\overset{\lambda}{R_x},\overset{\mu}{R_x}\right)$ we mean the mapping $\overset{\nu}{R_x}$ of C into itself by which $\overset{\nu}{R_x}(z)=\overset{\lambda}{R_x}(z)$ for $z \in A$, $\overset{\nu}{R_x}(z)=\overset{\mu}{R_x}(z)$ for $z \in B$. Especially the invariant mapping is denoted by E, and the mapping of A or B into only an element p is denoted by Z_p. We often denote $\overset{\lambda}{R_x}\overset{\lambda}{L_y}=\overset{\lambda}{L_y}\overset{\lambda}{R_x}$ by $\overset{\lambda}{R_x} \approx \overset{\lambda}{L_y}$ for short.

Proof. Since $\overset{\nu}{R}_p=(\overset{\lambda}{R}_p, E)$, $\overset{\nu}{L}_p=(\overset{\lambda}{L}_p, E)$ for $p \in A$, and $\overset{\nu}{R}_q=\left(Z_q, \overset{\mu}{R}_q\right)$, $\overset{\nu}{L}_q=\left(Z_q, \overset{\mu}{L}_q\right)$ for $q \in B$, we have immediately

$$\overset{\nu}{R}_p\overset{\nu}{L}_q=\left(\overset{\lambda}{R}_p, E\right)\left(Z_q, \overset{\mu}{L}_q\right)=\left(Z_q, \overset{\mu}{L}_q\right)=\left(Z_q, \overset{\mu}{L}_q\right)\left(\overset{\lambda}{R}_p, E\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p$$

Similarly $\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_q,\overset{\mu}{R}_q\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q$, $\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, E\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p$,

$$\overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_{q\mu q}, \overset{\mu}{R}_q\overset{\mu}{L}_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q .$$

Theorem 7. *Suppose that $A(\lambda)$ has a two-sided zero 0. If ν is defined as*

$$x \nu y=x \lambda y \quad \textit{for } x, y \in A , \quad x \nu y=x \mu y \quad \textit{for } x, y \in B,$$
$$x \nu y=y \nu x=0 \qquad \textit{for} \quad x \in A, y \in B,$$

then $C(\nu)$ is a semi-group.

Proof. Since $\overset{\nu}{R}_p=\left(R_p, Z_o\right)$, $\overset{\nu}{L}_p=\left(\overset{\lambda}{L}_p, Z_o\right)$ for $p \in A$, and $\overset{\nu}{R}_q=\left(Z_o, \overset{\mu}{R}_q\right)$, $L_q=\left(Z_o, \overset{\mu}{L}_q\right)$ for $q \in B$, we have

$$\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p, \qquad \overset{\nu}{R}_p\overset{\nu}{L}_q, =\left(Z_o, Z_o\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p,$$

$$\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_o, \ Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q, \qquad \overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_o, \overset{\mu}{R}_q\overset{\mu}{L}_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q.$$

Theorem 8 *Let $A(\lambda)$ include a two-sided zero 0 and let $B(\mu)$ be defined as $x\mu y=x$. If $C(\nu)$ is given as:*

$$x \nu y=x \lambda y \quad \text{for} \quad x, y \in A, \quad x \nu y=x \mu y \quad \textit{for } x, y \in B,$$
$$x \nu y=0 \quad \textit{for } x \in A, y \in B, \quad x \nu y=x \quad \textit{for } x \in B, y \in A,$$

then $C(\nu)$ is a semi-group.

Proof. $\overset{\nu}{R}_p=\left(\overset{\lambda}{R}_p, \ E\right)$, $L_p=\left(\overset{\lambda}{L}_p, Z_o\right)$, $\overset{\nu}{R}_q=\left(Z_o, E\right)$,[3] $\overset{\nu}{L}_q=\left(Z_q, Z_q\right)$ for $p \in A$, $q \in B$.

Then $\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p$, $\overset{\nu}{R}_p\overset{\nu}{L}_q=\left(Z_q, Z_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p$,

$$\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_o, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q, \qquad \overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_q, Z_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q .$$

Theorem 9 *Let $A(\lambda)$ be defined as $x \lambda y=y$. If ν is given as:*

$$x \nu y=r \quad (\textit{fixed} \in A) \quad \textit{for } x \in A, y \in B, \quad x \nu y=y \quad \textit{for } x \in B, \quad y \in A,$$
$$x \nu y=x \lambda y \quad \textit{for } x, y \in A, \quad x \nu y=x \mu y \quad \textit{for } x, y \in B,$$

then $C(\nu)$ is a semi-group.

Proof. $\overset{\nu}{R}_p=\left(Z_p, \ Z_p\right)$, $\overset{\nu}{L}_p=\left(E, Z_r\right)$,[4] $\overset{\nu}{R}_q=\left(Z_r, \overset{\mu}{R}_q\right)$, $L_q=\left(E, \overset{\mu}{L}_q\right)$ *for $p \in A$, $q \in B$.*

---

3) $\overset{\mu}{R}_q=E$, $\overset{\mu}{L}_q=Z_q$ *for $q \in B$.*

4) $\overset{\lambda}{L}_p=E$, $\overset{\lambda}{R}_p=Z_p$ *for $p \in A$.*

Then $\overset{\nu}{R}_p \overset{\nu}{L}_p = \left(Z_p,\ Z_p\right) = \overset{\nu}{L}_p \overset{\nu}{R}_p,\ \ \overset{\nu}{R}_p \overset{\nu}{L}_b = \left(Z_p,\ Z_p\right) = \overset{\nu}{L}_q \overset{\nu}{R}_q,$

$\overset{\nu}{R}_q \overset{\nu}{L}_p = \left(Z_r,\ Z_r\right) = \overset{\nu}{L}_p \overset{\nu}{R}_q,\quad \overset{\nu}{R}_q \overset{\nu}{L}_q = \left(Z_r,\ \overset{\mu}{R}_q \overset{\mu}{L}_q\right) = \overset{\nu}{L}_q \overset{\nu}{R}_q.$

As the special cases we consider the one-adjoined extension i. e., the semi-group A^* obtained by adjoining only an idempotent s to a semi-group A.

Corollary *Let* $C(\nu) = A(\lambda) \cup \{s\}$ *where* $s \bar{\in} A(\lambda)$. *If* ν *is given as follows,* $C(\nu)$ *is a semi-group in each cach case of* $(1) \sim (5)$.

(1) $x \nu y = x \lambda y$ *for* $x, y \in A$, $\quad x \nu s = s \nu x = x$ *for* $x \in A$, $\quad s \nu s = s.$

(2) $x \nu y = x \lambda y$ *for* $x, y \in A$, $\quad x \nu s = s \nu x = s$ *for* $x \in A$, $\quad s \nu s = s.$

(3) $A(\lambda)$ *has a two-sided zero 0.*

$x \nu y = x \lambda y$ *for* $x, y \in A$, $\quad x \nu s = s \nu x = 0$ *for* $x \in A$,

$s \nu s = s$

(4) $A(\lambda)$ *has a two-sided zero 0.*

$x \nu y = x \lambda y$ *for* $x, y \in A$, $\quad x \nu s = 0,$

$s \nu x = s$ *for* $x \in A$, $\quad s \nu s = s.$

(5) $A(\lambda)$ *is defined as* $x \lambda y = y$.

$x \nu y = x \lambda y$ *for* $x, y \in A$, $\quad x \nu s = p$ (*fixed* $\in A$) *for* $x \in A$,

$s \nu x = x$ *for* $x \in A$, $\quad s \nu s = s.$

Next, as to isomorphism between the same kind of one-adjoined extensions, we have

Theorem 10. *Let C and C′ be the same kind (1) or (2) of one-adjoined extensions of A and A′ repectively. C is isomorphic with C′ if and only if A is isomorphic with A′.*

Proof. Suppose C is isomorphic with C'. Let a and a' be units or zeros of C and C' respectively. Then by the uniqueness of a unit or zero we see that a is mapped to a'. Accordingly A is isomorphic with A'. The converse is clear.

Now we compose non-universal[5] one-adjoined extension of a given semi-group. Let $B(\mu) = A(\lambda) \cup \{s\}$ where $A(\lambda)$ is a semi-group and $s \bar{\in} A(\lambda)$.

Theorem 11. *If* μ *is defined as:*

$x \mu y = x \lambda y$ *for* $x, y \in A$, $x \mu s = x \lambda t$, $s \mu x = t \lambda x$ *for* $x \in A$, t(*fiexed*) $\in A$,

$s \mu s = t \lambda t,$

then $B(\mu)$ *is a semi-group.*

Proof. $\overset{\mu}{R}_p = \left(\overset{\lambda}{R}_p,\ t\lambda p\right)$,[6] $\overset{\mu}{L}_p = \left(\overset{\lambda}{L}_p,\ p\lambda t\right)$ *for* $p \in A$, and $\overset{\mu}{R}_s = \left(\overset{\lambda}{R}_t,\ t \lambda t\right)$, $\overset{\mu}{L}_s = \left(\overset{\lambda}{L}_t,\ t \lambda t\right)$

From them we readily have $\overset{\mu}{R}_p \approx \overset{\mu}{L}_p,\ \ \overset{\mu}{R}_p \approx \overset{\mu}{L}_s,\ \ \overset{\mu}{R}_s \approx \overset{\mu}{L}_p,\ \ \overset{\mu}{R}_s \approx \overset{\mu}{L}_s$

The following theorem is worth notice.

5) We mean by it that the one-adjoined extension $B(\mu)$ is a non-universal. See [1] with respect to "universal."

6) By $\overset{\mu}{R}_p = \left(\overset{\lambda}{R}_p,\ t \lambda p\right)$ we mean that $\overset{\mu}{R}_p(z) = \overset{\lambda}{R}_p(z)$ for $z \in A$, ond $\overset{\mu}{R}_p(s) = t \lambda p$.

Theorem 12. *If $A(\lambda)$ has a two-sided unit, the non-universal one-adjoined extensions are no other than ones above shown by Theorem 11.*

Proof. Suppose that $B(\mu)$ be the non-universal one-adjoined extension of $A(\lambda)$. Let a be a two-sided unit of $A(\lambda)$, and let $s\,\mu\,a=p$, $a\,\mu\,s=q$, and $s\,\mu\,s=u$. Then $\overset{\mu}{R_a}=\begin{pmatrix}E, & p\end{pmatrix}$, $\overset{\mu}{L_a}=\begin{pmatrix}E, & q\end{pmatrix}$, moreover we set $\overset{\mu}{R_s}=\begin{pmatrix}R'_s, & u\end{pmatrix}$, $\overset{\mu}{L_s}=\begin{pmatrix}L'_s, & u\end{pmatrix}$. Since $\overset{\mu}{R_a}\approx\overset{\mu}{L_a}$ according to Theorem 1, we see that $p=q$. On the other hand it follows from [5] that $\overset{\mu}{R_s}\overset{\mu}{R_p}=\overset{\mu}{R_a}$, $\overset{\mu}{L_s}\overset{\mu}{L_t}=\overset{\mu}{L_p}$, concluding that $\begin{pmatrix}R'_s, & u\end{pmatrix}=\begin{pmatrix}\overset{\lambda}{R_p}, & s\,\mu\,p\end{pmatrix}$, $\begin{pmatrix}L'_s, & u\end{pmatrix}=\begin{pmatrix}\overset{\lambda}{L_p}, & p\,\mu\,s\end{pmatrix}$, consequently $R'_s=\overset{\lambda}{R_p}$, $L'_s=\overset{\lambda}{L_p}$ and $u=p\,\mu\,s=s\,\mu\,p$. We get at once $u=p\lambda p$. The proof has been completed.

§ 3 Addendum.

For the preparation of § 4, 5, a few theorems will be added.

Theorem 13. *A finite semi-group has at least an idempotent* [6].

Theorem 14. *A finite semi-group S is a right (left) groupoid*[7] *if and only if $L_x \underset{i}{=} S$ ($R_x \underset{i}{=} S$) for every $x \in S$. Especially it is a group if and only if $R_x \underset{i}{=} S$ as well as $L_x \underset{i}{=} S$ for every $x \in S$.*

Theorem 15. *If the algebra S has an idempotent a and every L_x(or R_x)is either E or Z_a for every $x \in S$, then S is a semi-group.*

Proof of Theorem 15. We see that $R_a=Z_a$. Let us consider two cases: (1) $L_a=E$, (2) $L_a=Z_a$.

(1) When $L_a=E$, it follows that $L_x=E$ for every $x \in S$. This is out of the question [8].

(2). When $L_a=Z_a$, we see that $R_x(a)=a$ for all $x \in S$ and $R_x \approx L_x$ for every $x \in S$. Hence S is a semi-group by Theorem 1.

In the next two paragraphs, we shall determine all types of semigroups, up to isomorphism, defined in $\{a, b\}$ and in $\{a, b, c\}$.

§ 4. Semi-groups of Order 2.

We can see easily that the following 5 operations $\lambda_1 \sim \lambda_4$ and μ defined in $\{a, b\}$ are all semi-groups.[8]

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7) See [7].

8) We denote, for example, the table

	a	b
a	a	b
b	a	b

by

a	b
a	b

$$\begin{array}{|cc|}a & b\\ a & b\end{array}\qquad \begin{array}{|cc|}a & a\\ a & b\end{array}\qquad \begin{array}{|cc|}a & b\\ b & a\end{array}\qquad \begin{array}{|cc|}a & a\\ b & b\end{array}\qquad \begin{array}{|cc|}a & a\\ a & a\end{array}$$
$$\lambda_1 \qquad\quad \lambda_2 \qquad\quad \lambda_3 \qquad\quad \lambda_4 \qquad\quad \mu$$

In fact, λ_1, λ_4 and μ are semi-groups by Theorem 15, λ_2 by (3) of the Corollary, λ_3 by Theorem 14.

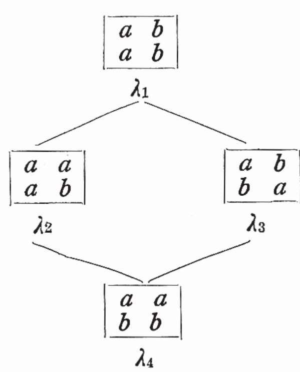
It can be proved, furthermore, that semi-groups of order 2 are nothing but these 5 types up to isomorphism. In order to prove this it is sufficient to discuss the following 3 types among all algebras which are possible to be given in $\{a, b\}$.

$$\begin{array}{cc}a & b\\ b & b\end{array}\qquad \begin{array}{cc}a & b\\ a & a\end{array}\qquad \begin{array}{cc}a & a\\ b & a\end{array}$$
$$\nu_1 \qquad\quad \nu_2 \qquad\quad \nu_3$$

Though ν_1 is isomorphic to λ_2, ν_2 is not a semi-group, neither ν_3, because $R_b \neq L_b$.

Let us now study the ordering in the universal semi-group system. By Theorem 2, we see $\lambda_2 \not\gtrless \lambda_3$; and $\lambda_1 \underset{\not<}{\geqq} \lambda_2$, $\lambda_1 \underset{\not<}{\geqq} \lambda_3$ dually $\lambda_4 \underset{\not>}{\leqq} \lambda_2$, $\lambda_4 \underset{\not>}{\leqq} \lambda_3$.

The diagram of the universal semi-group system of order 2 is as follows:



where λ_1 is a right groupoid, λ_4 a left groupoid, λ_3 a group, λ_2 a semi-lattice.

§ 5. Semi-groups of order 3.

1. *Non-universal Semi-groups.*

Without loss of generality, it may be assumed that c does not belong to the value range[9] of $S=\{a, b, c\}$, and $\{a, b\}$ is a sub-semi-group of S; and so all the types of semi-groups $\{a, b\}$ are as follows up to isomorphism or anti- isomorphism.

$$\begin{array}{cc}a & a\\ a & a\\ \multicolumn{2}{c}{(1)}\end{array}\qquad \begin{array}{cc}a & b\\ a & b\\ \multicolumn{2}{c}{(2)}\end{array}\qquad \begin{array}{cc}a & a\\ a & b\\ \multicolumn{2}{c}{(3)}\end{array}\qquad \begin{array}{cc}a & b\\ b & a\\ \multicolumn{2}{c}{(4)}\end{array}$$

Now, we shall discuss (1)∼(4) successively.

---

9) By the value range A^* of the subset A we mean the set composed of elements $z=xy$ for $x, y \in A$.

(1) $ac=ca=a$ follows from $R_c \approx L_a$, $L_c \approx R_a$; $bc=cb=a$ from $R_b \approx L_c$, $L_b \approx R_c$ (Theorem 1).

we have

$$\mu_1: \left|\begin{array}{ccc} a & a & a \\ a & a & a \\ a & a & a \end{array}\right| \qquad \mu_2: \left|\begin{array}{ccc} a & a & a \\ a & a & a \\ a & a & b \end{array}\right|$$

(2) From Theorem 1 and 4, at once $ca=a$, $cb=b$; and we get

$$\mu_3: \left|\begin{array}{ccc} a & b & a \\ a & b & a \\ a & b & a \end{array}\right|$$

with which another is isomorphic.

The above μ_1, μ_2 and μ_3 prove to be semi-groups directly from Theorem 1.

(3) (4) By Theorem 11 and 12, we have

$$\mu_4: \left|\begin{array}{ccc} a & a & a \\ a & b & a \\ a & a & a \end{array}\right| \qquad \mu_5: \left|\begin{array}{ccc} a & a & a \\ a & b & b \\ a & b & b \end{array}\right| \qquad \mu_6: \left|\begin{array}{ccc} a & b & a \\ b & a & b \\ a & b & a \end{array}\right| \qquad \mu_7: \left|\begin{array}{ccc} a & b & b \\ b & a & a \\ b & a & a \end{array}\right|$$

Moreover, adding

$$\mu_8: \left|\begin{array}{ccc} a & a & a \\ b & b & b \\ a & a & a \end{array}\right|$$

which is anti-isomorphic with μ_3, we have obtained all non-universal semi-groups $\mu_1 \sim \mu_8$.

2 *Universal Semi-groups.*

Without loss of generality, R_a and L_a may be assumed only as follows:

(1)	$L_a=(a, b, c)$,	$R_a=(a, b, c)$, [10]
(2)	$L_a=(a, b, c)$,	$R_a=(a, b, b)$,
(3)	$L_a=(a, b, c)$,	$R_a=(a, b, a)$,
(4)	$L_a=(a, b, c)$,	$R_a=(a, a, a)$,
(5)	$L_a=(a, a, a)$,	$R_a=(a, b, b)$,
(6)	$L_a=(a, a, a)$,	$R_a=(a, b, a)$,
(7)	$L_a=(a, b, a)$,	$R_a=(a, b, a)$,
(8)	$L_a=(a, b, b)$,	$R_a=(a, b, b)$,
(9)	$L_a=(a, a, a)$,	$R_a=(a, a, a)$,
(10)	$L_a=(a, b, a)$,	$R_a=(a, a, c)$.

Because it is necessary that $R_a \approx L_a$; and the others are isomorphic or anti-isomorphic with one of the above by the mapping $\begin{pmatrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a & c & b \end{pmatrix}$. Now, we denote by $[b, c]$ the value range of the subset $\{b, c\}$. Successively the cases (1)$\sim$(10) will

10) By (a, b, a), for example, we mean the mapping $\begin{pmatrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a & b & a \end{pmatrix}$.

be discussed.

(1) When $a \mathrel{\bar{\in}} [b, c]$, we have from (1) of Corollary and Theorem 10

$$\underset{\lambda_1}{\left|\begin{matrix} a & b & c \\ b & b & b \\ c & b & b \end{matrix}\right|} \qquad \underset{\lambda_2}{\left|\begin{matrix} a & b & c \\ b & b & c \\ c & b & c \end{matrix}\right|} \qquad \underset{\lambda_3}{\left|\begin{matrix} a & b & c \\ b & b & b \\ c & b & c \end{matrix}\right|} \qquad \underset{\lambda_4}{\left|\begin{matrix} a & b & c \\ b & b & c \\ c & c & b \end{matrix}\right|} \qquad \underset{\lambda_5}{\left|\begin{matrix} a & b & c \\ b & b & b \\ c & c & c \end{matrix}\right|}$$

When $a \in [b, c]$, we can suppose $cb=a$ or $cc=a$, to which others are mapped. If $cb=a$, then by Theorem 3, we get (group)

$$\underset{\lambda_6}{\left|\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix}\right|}$$

If $cc=a$, we see that $cb=bc=b$ by Theorem 3, and $bb=b$ from $R_c \approx L_b$ (Theorem 1).

$$\underset{\lambda_7}{\left|\begin{matrix} a & b & c \\ b & b & b \\ c & b & a \end{matrix}\right|}$$

(2) By Theorem 1 and 3, $a \mathrel{\bar{\in}} [b, c]$; considering (4) of Theorem 4 and $R_b \subset R_a$ we have

$$\underset{\lambda_8}{\left|\begin{matrix} a & b & c \\ b & b & b \\ b & b & b \end{matrix}\right|} \qquad \underset{\lambda_9}{\left|\begin{matrix} a & b & c \\ b & b & c \\ b & b & c \end{matrix}\right|}$$

(3) By (4) of Theorem 4, $L_c=(a, b, c)$; we get $bc=b$ from $R_a \approx L_b$ and $bb=b$ from $R_c \approx L_b$, $R_b \subset R_a$.

$$\underset{\lambda_{10}}{\left|\begin{matrix} a & b & c \\ b & b & b \\ a & b & c \end{matrix}\right|}$$

(4) See the proof of Theorem 15

$$\underset{\lambda_{11}}{\left|\begin{matrix} a & b & c \\ a & b & c \\ a & b & c \end{matrix}\right|}$$

(5) & (6) By (3) of Theorem 4, it holds $L_b=(b, b, b)$; and so either b or c is a right-unit. Hence we have semi-groups each of which is isomorphic with one belonging to (1)$\sim$(4).

(7) From Theorem 3 follows that neither R_b nor L_b contains c; hence $cc=c$. On the other hand, $b \in R_c$, $b \in L_c$, that is, $bc=cb=b$, showing that c is a unit. Therefore this case is reduced to the previous one.

(8) Similarly $cc=c$. From this it concludes that $bc=cb=b$, because we require $R_a \approx L_c$, $R_c \approx L_a$. If $bb=a$, then $R_a \not\approx L_b$. Hence $bb=b$; we have

$$\underset{\lambda_{12}}{\left|\begin{matrix} a & b & b \\ b & b & b \\ b & b & c \end{matrix}\right|}$$

(9) Let us investigate the case that a semi-group has no idempotent but a and the value range $[b, c]$ contains a. For, if $a \bar{\in} [b, c]$, then we have from (2) of Corollary and Theorem 10

$$\begin{array}{ccc} a & a & a \\ a & b & c \\ a & b & c \end{array} \qquad \begin{array}{ccc} a & a & a \\ a & b & b \\ a & b & c \end{array} \qquad \begin{array}{ccc} a & a & a \\ a & b & c \\ a & c & b \end{array} \qquad \left|\begin{array}{ccc} a & a & a \\ a & b & b \\ a & c & c \end{array}\right|$$

$$\lambda'_{10} \qquad\qquad \lambda'_{3} \qquad\qquad \lambda'_{7} \qquad\qquad \lambda_{13}$$

whereas λ'_{10} is isomorphic with λ_{10}, λ'_3 with λ_3, λ'_7 with λ_7; and λ_{13} is anti-isomorphic with λ_{10}.

If one at least of b and c is idempotent, then the semi-group is isomorphic with one of (1)$\sim$(8). Now we take up only the following cases, which are all out of our consideration.

$$\begin{array}{ccc} a & a & a \\ a & a & \\ a & & a \end{array} \qquad \begin{array}{ccc} a & a & a \\ a & a & \\ a & & b \end{array} \qquad \begin{array}{ccc} a & a & a \\ a & c & \\ a & & b \end{array}$$

$$\text{i)} \qquad\qquad \text{ii)} \qquad\qquad \text{iii)}$$

i) The element cb must be either b or c, but whatever cb is, $R_b \not\approx L_c$.

ii) Either cb or bc must be c. Then $R_b \not\approx L_c$ or $L_b \not\approx R_c$.

iii) By the assumption, $cb=a$ or $bc=a$. However it follows that $R_b \not\approx L_c$ or $L_b \not\approx R_c$.

10) It follows from $L_a \approx R_b$ that $cb=b$, contradicting to $R_a \approx L_c$. Hence there is none with $L_a=(a, b, a)$, $R_a=(a, a, c)$.

In addition to $\lambda_1 \sim \lambda_{13}$, we have the remaining ones which are anti-isomorphic with the former.

$$\left|\begin{array}{ccc} a & a & a \\ b & b & b \\ c & c & c \end{array}\right| \qquad \left|\begin{array}{ccc} a & b & b \\ b & b & b \\ c & b & b \end{array}\right| \qquad \left|\begin{array}{ccc} a & b & b \\ b & b & b \\ c & c & c \end{array}\right|$$

$$\lambda_{14} \qquad\qquad \lambda_{15} \qquad\qquad \lambda_{16}$$

We can easily see that $\lambda_1 \sim \lambda_{16}$ thus obtained are semi-groups which are not isomorphic each other.

3 *The Ordering of the Universal Semi-group System*

At first we define a term as following. If the system $\mathfrak{N}$ of universal semi-group operations defined in a set S satisfies the conditions (1) and (2) as follows, $\mathfrak{N}$ is called the normal represent system of universal semi-groups with respect to S.

(1) For any $\lambda, \mu \in \mathfrak{N}$ $(\lambda \neq \mu)$, one is not isomorphic with the other.

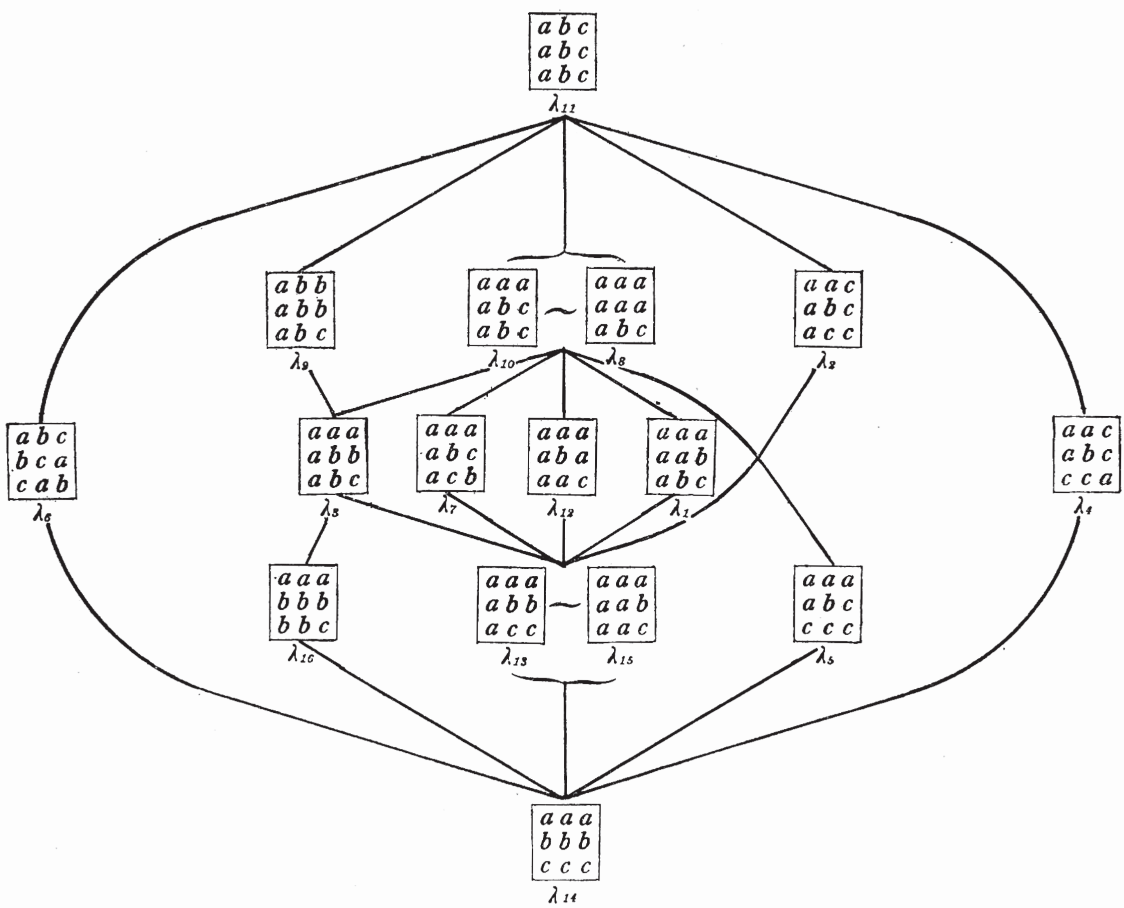
(2) For any $\lambda \in \mathfrak{N}$, $\mathfrak{N}$ contains ν which is identically anti-isomorphic with λ, that is, $\nu=\lambda^{\mathfrak{e}}$ [10] where $\mathfrak{e}$ is an identical translation on S.

For example, we have as the normal represent system of universal semi-groups:

$\begin{array}{ccc} a & a & a \\ a & a & b \\ a & b & c \end{array}$	$\begin{array}{ccc} a & a & c \\ a & b & c \\ a & c & c \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & b \\ a & b & c \end{array}$	$\begin{array}{ccc} a & a & c \\ a & b & c \\ c & c & a \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & c \\ c & c & c \end{array}$	$\begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & c \\ a & c & b \end{array}$	$\begin{array}{ccc} a & a & a \\ a & a & a \\ a & b & c \end{array}$
λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
$\begin{array}{ccc} a & b & b \\ a & b & b \\ a & b & c \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & c \\ a & b & c \end{array}$	$\begin{array}{ccc} a & b & c \\ a & b & c \\ a & b & c \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & a \\ a & a & c \end{array}$	$\begin{array}{ccc} a & a & a \\ a & b & b \\ a & c & c \end{array}$	$\begin{array}{ccc} a & a & a \\ b & b & b \\ c & c & c \end{array}$	$\begin{array}{ccc} a & a & a \\ a & a & b \\ a & a & c \end{array}$	$\begin{array}{ccc} a & a & a \\ b & b & b \\ b & b & c \end{array}$
λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}

where these λ_i are isomorphic with the previously written λ_i.

The diagram of the system is obtained by Theorem 2 or [9].



As easily seen, this system forms a lattice, but the lattice depends on the represent system.

References.

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ma University. Vol. 1, (1950) p. 37.

[2] T. TAMURA, *On the system of semigroup operations defined in a set*, Journal of the Gakugei Faculty, Tokushima Unlversity, (natural science) Vol. 2, (1952) pp. 2-3.

[3] The paper [2] pp. 2-6.

[4] The paper [2] p. 3.

[5] The paper [2] p. 2

[6] D. REES, *On semi-groups*, Proceedings of the Cambridge Philosophical Society, Vol. 36, pt. 4 (1940) § 1. 2.

[7] The paper [1] pp. 38-39.

[8] The paper [2] p. 4.

[9] See [2].

[10] The paper [2] p. 8.

The operation λ^{e} is defined as: $x \lambda^{e} y = y \lambda x$.

Addendum.

It is regret that I can not refer to the papers by Clifford, Suschkewitsch, etc., for lack of literature in our university. I fear that some of our results may have been contained in a study by someone.

Notes on General Analysis (II)

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In these notes, we shall first state of the necessary and sufficient conditions that a function should be homogeneous polynomials of degree n, in § 1. In § 2, we shall investigate whether some of the theorems of Schwarz on regular functions of complex variables will be able to be extended to the case of functions whose domain and range both lie in complex-Banach-spaces or not. Finally, in § 3, we shall investigate the state of the boundary of the domain $G(h_k)$.

§ 1. Homogeneous polynomials.

Let E and E' be two complex-Banach-spaces.

Definition 1.*) *An E'-valued function $x'=p(x)$ defined on E is called a homogeneous polynomial of degree n, if the following conditions are satisfied: (1) $p(x)$ is strongly continuous at each point of E, (2) for each x and y in E, and for any complex number a, $p(x+ay)$ can be expressed as $p(x+ay)=\sum_{k=0}^{n} p_k(x,y)a^k$, where $P_k(x,y)$ are arbitrary E'-valued functions of two variables x and y, (3) $P_n(x,y)\neq O$ for some x and y, (4) $p(ax)=a^n p(x)$.*

Definition 2.*) *An E'-valued function $x'=f(x)$ defined on a domain D of E is called analytic, if it is strongly continuous and G-differentiable on D.*

Theorem 1. *The necessary and sufficient conditions that $p(x)$ should be a homogeneous polynomial of degree n are that it is analytic on E and satisfies $p(ax)=a^n p(x)$.*

Proof. If p(x) is a homogeneous polynomial of degree n, it satisfies $p(\alpha x)=\alpha^n p(x)$ by (4) and is strongly continuous by (1). The condition (2) shows that p(x) is G-differentiable at any point in E. Thus we see that the conditions are necessary. Conversely, let p(x) be analytic at any point of E and satisfies $p(\alpha x)=\alpha^n p(x)$. Then we have

$$P(x+\alpha y)=\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi-\alpha}\,d\xi=\frac{1}{2\pi i}\int_c\left(\sum_{m=0}^{\infty}\frac{P(x+\xi y)}{\xi^{m+1}}\alpha^m\right)d\xi$$

$$=\sum_{m=0}^{\infty}\frac{I}{2\pi i}\int_c\frac{P(x+\xi y)}{\xi^{m+1}}\alpha^m d\xi$$

$$=\sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_c\frac{P(x+\xi y)}{\xi^{m+1}}d\xi\right)\alpha^m ,$$

where C is a circle of radius r and $r>|\alpha|$.
While, $p(\alpha x)=\alpha^n p(x)$, we have

$$\frac{1}{2\pi i}\int_c\frac{P(x+\xi y)}{\xi^{m+1}}\,d\xi=\frac{1}{2\pi i}\int_c\frac{P((1/\xi)x+y)}{\xi^{m+1-n}}\,d\xi$$

Put $\frac{1}{\xi}=\eta$, then $d\xi=-\frac{1}{\eta^2}d\eta$ and $\frac{1}{\xi^{m+1-n}}=\eta^{m-n+1}$

Therefore, we have

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{n+1}}\, d\xi = -\frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\, d\eta\ .$$

Since the left side integral is taken counterclockwise along the circle $|\xi|=r$, the right side integral is taken clockwise along the circle $|\eta|=\frac{1}{r}$. Integrating counterclockwise along the circle $|\eta|=\frac{1}{r}$, we have

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\, d\xi = \frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\, d\eta\ .$$

Since $p(\eta x+y)$ is regular on $|\eta|\leqq\frac{1}{r}$,

$$\frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\, d\eta = P(y)\,,\ \text{ when }\ m=n\,,$$

$$=0\,,\ \text{ for }\ m>n+1\,.$$

Put

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\, d\xi = P_m(x,y)\,,$$

then we have

$$P(x+\alpha y) = \sum_{m=0}^{n} P_m(x,y)\alpha^m\,.$$

Since $p_n(x, y)=p(y)$, $p_n(x, y)\not\equiv 0$. This completes the proof.

Corollary. *The necessary and sufficient condition that $\mu(x)$ should be linear is that $\mu(x)$ is analytic on E and satisfies $\mu(ax)=a\mu(x)$.*

§ 2. Extension of the Schwarz's theorem.

The purpose of this chapter is to extend the Schwarz's theorem of complex variables to the case of complex-Banach-spaces. The theorem of Schwarz is described as follows: If f(z) is regular in the circle $|z|<R$ and satisfies $f(0)=0$ and $|f(z)|\leqq M$ in the circle $|z|<R$, then $|f(z)|\leqq\frac{M}{R}|z|$. If the equality is established at a point of $|z|<R$, then $f(z)\equiv\frac{M}{R}e^{i\theta}z$.

This theorem is not always true in our cases.

Theorem 2. *Let an E'-valued function $f(x)$ defined in the sphere $\|x\|<R$ be analytic and satisfies $f(0)=0$ and $\|f(x)\|\leqq M$ in the sphere $\|x\|<R$. Then $\|f(x)\|\leqq\frac{M}{R}\|x\|$*

Proof. Since f(x) is analytic in the sphere $\|x\|<R$ and $f(0)=0$, we have

$$f(x) = \sum_{n=1}^{\infty} h_n(x)\,, \qquad \cdots\cdots(1)$$

for an arbitrary x in $\|x\|<R$, where $h_n(x)$ is a homogeneous polynomial of degree

n. Now, we fix x in $\|x\|<R$. From (1), we have

$$f(\alpha x)=\sum_{n=1}^{\infty} h_n(x)\alpha^n .$$

$f(\alpha x)$ is analytic about α, when $|\alpha|<\frac{R}{\|x\|}$, where clearly $\frac{R}{\|x\|}>1$. Since $f(\alpha x)$ is an analytic function of α, $\frac{f(\alpha x)}{\alpha}=\sum_{n=1}^{\infty}h_n(x)\alpha^{n-1}$ is also analytic in the circle $|\alpha|<\frac{R}{\|x\|}$. Let r be an arbitrary positive number which satisfies $r<\frac{R}{\|x\|}$, then $\|\frac{f(\alpha x)}{\alpha}\|\leqq\frac{M}{r}$, when $|\alpha|=r$, because $\|f(\alpha x)\|\leqq M$ and $|\alpha|=r$. Since $\|\frac{f(\alpha x)}{\alpha}\|$ is subharmonic as to α, $\|\frac{f(\alpha x)}{\alpha}\|$ takes its maximum on $\|\alpha\|=r$. Thus we see that $\|\frac{f(\alpha x)}{\alpha}\|\leqq\frac{M}{r}$, for $|\alpha|\leqq r$. Since r is an arbitrary positive number satisfying $r<\frac{R}{\|x\|}$, we have $\|\frac{f(\alpha x)}{\alpha}\|\leqq M\Big/\frac{R}{\|x\|}=\frac{M}{R}\|x\|$,

for $|\alpha|<\frac{R}{\|x\|}$ (>1). Put $\alpha=1$, and we have

$$\|f(x)\|\leqq\frac{M}{R}\|x\| . \qquad\qquad (2)$$

Since x is an arbitrary point in $\|x\|<R$, (2) is held for $\|x\|<R$. This completes the proof.

In concluding this paragraph, we shall aford an example f(x) which satisfies following conditions

(1) $f(0)=0$,
(2) f(x) is analytic on $\|x\|<1$,
(3) $\|f(x)\|\leqq M$ on $\|x\|<1$,
(4) $\|f(x)\|=M\|x\|$ for some points in $\|x\|<1$,

and yet $\|f(x)\|\neq M\|x\|$

Let $X=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$ be a matrix of (2, 2)-type of complex numbers, and $\|X\|=Max(|x_{11}|, |x_{12}|, |x_{21}|, |x_{22}|)$. Then the set of such X is clearly complex-Banach-spaces. Put $\mu(X)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}X$ where $\infty>a>b>c>d>0$, and $M=a+b$. Then

$$\mu(X)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}=\begin{pmatrix}a x_{11}+b x_{21} & a x_{12}+b x_{22}\\ c x_{11}+d x_{21} & c x_{12}+d x_{22}\end{pmatrix} .$$

Clearly $\mu(X)$ is a linear function and we see that $\mu(o)=0$ and $\mu(X)$ is an analytic function on whole spaces by Corollary of Theorem 1. Since $\operatorname{Sup}_{\|X\|=1}\|\mu(X)\|=a+b=M$, we have $\|\mu(X)\|\leqq M$, when $\|X\|\leqq 1$. Put $X_1=\begin{pmatrix}\lambda & o\\ \lambda & \lambda\end{pmatrix}$ and $X_2=\begin{pmatrix}\lambda & o\\ o & \lambda\end{pmatrix}$, where $0<\lambda<1$.

Then $\|X_1\|=\lambda$ and $\|X_2\|=\lambda$. Since

$$\mu(X_1)==\begin{pmatrix} a\lambda+b\lambda & b\lambda \\ c\lambda+d\lambda & d\lambda \end{pmatrix}, \quad \|\mu(X_1)\|=\lambda(a+b)=M\cdot\|X_1\|.$$

While $\mu(X_2)=\begin{pmatrix} a\lambda & b\lambda \\ c\lambda & d\lambda \end{pmatrix}$ and we see that

$$\|\mu(X_2)\|=\lambda a<\lambda(a+b)=\|X_2\|\cdot M.$$

§ 3. On the boundary of G(hn)**).

Definition 3. *$G(h_n)$ is the interior of the region of convergence of a power series $\sum_{n=0}^{\infty} h_n(x)$.*

Definition 4. *Let x be an arbitrary point on $\|x\|=1$. $R(x)$ is the upper-bound of $|\alpha|$, for which $\sum_{n=0}^{\infty} h_n(\alpha x)$ is convergent and analytic at αx.*

Theorem 3. *If $|\alpha|=R(x)$, αx is the boundary point of $G(h_n)$.*

Proof. Since $\sum_{n=0}^{\infty} h_n(x)$ is analytic in $G(h_n)$**), $\sum_{n=0}^{\infty} h_n(x)$ is analytic at αx while αx lies in $G(h_n)$***), where $\|x\|=1$. But $\sum_{n=0}^{\infty} h_n(\alpha x)$ is not analytic when αx lies beyond $G(h_n)$, because $\sum_{n=0}^{\infty} h_n(x)$ does not always converge in any neighbourhood of αx. This proves that αx is a boundary point of $G(h_n)$.

Theorem 4. *$R(x)$ is lower semi-continuous on $\|x\|=1$.*

Proof. If $R(x)$ is not lower semi-continuous at a point x_0 on $\|x\|=1$, there exists a sequence $\{x_i\}$ such that x_i tends to x_0 and satisfies

$$R(x_i) < R(x_0)-\epsilon \qquad (i=1,2,3,\ldots),$$

for a suitable positive number ϵ. While, if $|\alpha|=R(x_i)$, there exists at least a point α_i on $|\alpha|=R(x_i)$ such that $\alpha_i x_i$ is a singular point of $\sum_{n=0}^{\infty} h_n(x)$. Since $|\alpha_i|=R(x_i)<R(x_0)-\epsilon$, $\{\alpha_i\}$ has at least a limiting point α_0. Then we have a subsequence $\{\alpha_{i'}\}$ of $\{\alpha_i\}$ which converges to α_0. Thus we see that $\alpha_{i'}x_{i'}$ converges to $\alpha_0 x_0$. Since $\alpha_0 x_0$ is a limiting point of singular points $\alpha_{i'}x_{i'}$, $\alpha_0 x_0$ is also a singular point of $\sum_{n=0}^{\infty} h_n(x)$. Since $|\alpha_i|<R(X_0)-\epsilon$, $|\alpha_0|\leq R(X_0)-\epsilon$.

This contradicts that $\sum_{n=0}^{\infty} h_n(x)$ is analytic at αx_0, when $|\alpha|<R(x_0)$.

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*) A. E. TAYLOR: (1) Analytic functions in general analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. 11 Vol. vl (1938). (2) Additions to the theory of polynomials in normed linear spaces (Tohoku M. J. 44. 1938). (3) On the properties of analytic functions in abstract spaces, Math. Ann. 115, 1938.

**) E. HILLE: Functional analysis and semi-groups. p. 85.

***) See Theorem 4. 7. 1 (HILLE, Functional analysis and semigroup, page 85). If G(hn) is non-void, then G(hn) is a c-convex c-star about θ. That is, if $X \in G(hn)$, then $\alpha X \in G(hn)$, where $|\alpha| \leq 1$.

Ueber die Verschiebung der Nullstellen einiger Funktionen, welche aus Integration gebrochener Ordnung hervorgeht.

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Es sei $f(z)$ eine analytische Funktion der komplexen Variablen z und α reell positiv. Man hat das Riemann-Liouvillesche Integral

$$J^\alpha f(z) = \int_0^z \frac{(z-u)^{\alpha-1}}{\Gamma(\alpha)} f(u)du\,,$$

das für gebrochenes α allgemein mehrdeutig ist. Betrachten wir nun

$$g(z) = z^{-\alpha} J^\alpha f(z) \equiv H^\alpha f(z)\,,$$

so erhalten wir eine analytische Funktion, deren Existenzgebiet eben mit demselben von $f(z)$ übereinstimmt. Gestattet nämlich $f(z)$ eine Taylorsche Entwicklung um $z=0$

$$f(z) = \sum_{n=0}^{\infty} c_n z^n/\Gamma(n+1)\,,$$

so lautet

$$g(z) = \sum_{n=0}^{\infty} c_n z^n/\Gamma(n+\alpha+1)\,.$$

Dafür aber gilt

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|/\Gamma(n+1)} = \overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|/\Gamma(n+\alpha+1)}\,,$$

und folglich besitzen beide Reihen dieselben Konvergenz- bezw. Fortsetzbarkeit-Eigenschaften. Inzwischen veranläßt unsere Verfahren gewißermaßen eine Veränderung der Nullstellen-Verteilung. Ist beispielsweise $f(z)=e^z$, so ergibt sich

$$g(z)=H^\alpha e^z=\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} = \frac{1}{\Gamma(\alpha+1)}\left\{1+\frac{z}{\alpha+1}+\frac{z^2}{(\alpha+1)(\alpha+2)}+\cdots\cdots\right\}.$$

Diese Funktion wird zwar wieder ganz transcendental, aber doch verschwindet in gewißen Punkten der Halbebene $\Re z \gtrless \alpha-1$, je nachdem $\alpha>1$ oder $1>\alpha>0$ ist, wie ich unten elementarweise zeigen werde(1).

§ **1.** Eine gerade Rechnung liefert sofort

$$H^\alpha e^z = \frac{1}{z^\alpha} J^\alpha e^z = \frac{1}{z^\alpha}\int_0^z \frac{(z-\zeta)^{\alpha-1}}{\Gamma(\alpha)} e^\zeta d\zeta = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} \quad (\alpha>0)\,. \qquad (1)$$

(1) Man vergleiche für etwas ausgearbeiteten Beweis, Pólya und Szegö, Aufgaben und Lehrsätze aus der Analysis, Bd. II S. 70, S. 260.

Ins besondere gilt für $\alpha=1$

$$He^z=\frac{1}{z}(e^z-1)\,,$$

was in den auf der imaginären Achse liegenden Punkten $z=2n\pi i$ verschwindet, wo n ganz $\neq 0$ ist. Auch für jedes reelle x ist im allgemeinen

$$H^\alpha e^x=\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}e^{xt}dt>0\,,$$

so daß auf der reellen Achse es keine Nullstelle gibt. Anderseits, falls $z_0=r_0\exp i\theta_0$ eine Nullstelle ist, so muß $\bar{z}_0=r_0\exp(-i\theta_0)$ auch dieselbe sein. Deshalb darf ich unten mich nur auf den Fall beschränken, worin $0<\theta(=\arg z)<\pi$, d. h. der imaginäre Teil $y>0$ ist.

Setzt man nun in (1) $z=re^{i\theta}$, $\zeta=\rho e^{i\theta}$ (θ fest), so erhält man

$$H^\alpha e^z=\frac{1}{r^\alpha}\int_0^r\frac{(r-\rho)^{\alpha-1}}{\Gamma(\alpha)}\exp\{\rho(\cos\theta+i\sin\theta)\}\,d\rho\,,$$

oder, indem man nochmals $\rho\sin\theta=\eta$, $r\sin\theta=y$, $\cot\theta=c$ einführt,

$$\begin{aligned}H^\alpha e^z&=\frac{1}{y^\alpha}\int_0^y\frac{(y-\eta)^{\alpha-1}}{\Gamma(\alpha)}e^{c\eta}(\cos\eta+i\sin\eta)\,d\eta\\&=U+iV=W.\end{aligned}\tag{2}$$

Die hier in dem Integranden auftretende Funktion $(y-\eta)^{\alpha-1}e^{c\eta}=Y$ ist non-negativ im Intervalle $0\leqq\eta\leqq y$, und erlaubt für festes y

$$\frac{\partial Y}{\partial\eta}=(y-\eta)^{\alpha-2}e^{c\eta}\Big[\Re(z-\zeta)-(\alpha-1)\Big]\,,\quad\text{wo}\quad\Big|\Re(z-\zeta)\Big|\leqq\Big|\Re(z)\Big|\quad\text{ist.}$$

Hieraus fließt die Folgerung, daß für $\alpha>1$ bei $\Re z<\alpha-1$, $\frac{\partial Y}{\partial\eta}<0$ gilt und $Y(\geqq 0)$ nimmt stets ab, wenn η zunimmt, während für $0<\alpha<1$ bei $\Re z>\alpha-1$, $\frac{\partial Y}{\partial\eta}>0$ und $Y(>0)$ monoton zunimmt.

§ 2. Falls zuerst $\alpha>1$, $\Re z=x<\alpha-1$, so daß Y monoton fallend ist, schreibe ich

$$\begin{aligned}V=\Im H^\alpha e^z&=\frac{1}{y^\alpha}\int_0^y\frac{(y-\eta)^{\alpha-1}}{\Gamma(\alpha)}e^{c\eta}\sin\eta\,d\eta=\frac{1}{y^\alpha\Gamma(\alpha)}\int_0^y Y\sin\eta\,d\eta\\&=\int_0^\pi+\int_\pi^{2\pi}+\cdots\cdots+\int_{p\pi}^y\quad\text{mit}\quad y=(p+\varepsilon)\pi\,,\quad 0\leqq\varepsilon<1\\&=v_1-v_2+\cdots\cdots\cdots+(-1)^{p-1}v_p+(-1)^p v'_{p+1},\end{aligned}$$

wo $v_1>v_2>\cdots>v_p>v_p{}'_{+1}\geqq 0$ sind. Daher gelten, sowohl für $p=2q-1$ als $p=2q$

$$\begin{aligned}\int_0^y&=(v_1-v_2)+(v_3-v_4)+\cdots\cdots+(v_{2q-1}-v'_{2q})>0\\&=(v_1-v_2)+(v_3-v_4)+\cdots\cdots+(v_{2q-1}-v_{2q})+v'_{2q+1}>0\,.\end{aligned}$$

In diesem Falle ist also in der Linke der Gerade $x=\alpha-1(<0)$ durchaus $\Im H^\alpha e^z>0$, somit es dort keine Nullstelle gibt. Aber bei $\alpha=1$ zwar verschwindet $W=U(x, y, \alpha)+iV(x, y, \alpha)$ im Punkt $P(x=0, y=2n\pi, n\neq 0)$, während in demselben Punkte die Determinante

$$D=\begin{vmatrix}\frac{\partial U}{\partial x} & \frac{\partial U}{\partial y}\\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{\partial U}{\partial x} & -\frac{\partial V}{\partial x}\\ \frac{\partial V}{\partial x} & \frac{\partial U}{\partial x}\end{vmatrix}=\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2$$

den positiven Wert $1/4n^2\pi^2$ annimmt. Es leuchtet hieraus wegen eines wohlbekannten Satzes ein, daß durch jeden Punkt $P(0, 2n\pi)$ in der z-Ebene je eine Kurve $U(x,y,\alpha)=0$, $V(x,y,\alpha=0$ existiert, und daß für $\alpha<1$ alle diese in der Rechte der Gerade $x=\alpha-1$ liegen sollen.

§ 3. Um den Falle $1>\alpha>0$ zu behandeln, schicke ich folgende triviale Lemata voraus:

Lemma 1. Mit der konstanten Summe $2c$ zweier positiver Zahlen a und b wird das Produkt ab maximal bei $a=b=c$. Das Produkt ist je größer mit je kleiner Differenz; d.h. wenn $a_1+b_1=a_2+b_2=2c$ und $|a_1-b_1|<|a_2-b_2|$ sind, so besteht $a_1b_1>a_2b_2$. Dies ist klar, wegen der Identität $ab=\frac{1}{4}[(a+b)^2-(a-b)^2]=c^2-\frac{1}{4}(a-b)^2$.

Lemma 2. Sind a_1, b_1, a_2, b_2 alle positiv und $a_1b_1>a_2b_2$ sowie $|a_1-b_1|>|a_2-b_2|$, so folgt $(a_1+b_1)^2>(a_2+b_2)^2$ und deshalb auch $a_1+b_1>a_2+b_2$.

Da nun für $1>\alpha>0$, $x>\alpha-1$, die Funktion Y monoton wächst, gilt folgendes:

$$U=\Re H^\alpha e^z=\frac{1}{y^\alpha\Gamma(\alpha)}\int_0^y Y\cos\eta\,d\eta$$

$$=\int_0^\pi+\int_\pi^{2\pi}+\cdots\cdots+\int_{(p-1)\pi}^{p\pi}+\int_{p\pi}^{y}\qquad \text{mit } y=(p+\varepsilon)\pi\ ,\ \ 0\leqq\varepsilon<1$$

$$=-v_1+v_2-\cdots\cdots+(-1)^p v_p+(-1)^{p+1}v'_{p+1}\ ,$$

wo offenbar $v_1, v_2,\cdots\cdots,v_p>0$ und $v'_{p+1}\gtrless0$ sind. Überdies ist die Folge $v_1, v_2,\cdots\cdots$ sogar monoton wachsend. Zum Beweis dafür setzt man zuerst

$$\eta=h\pi+\tau\quad(0<\tau<2\pi),\ \ y-h\pi=t>2\pi\quad\text{und}\quad 1-\alpha=\beta>0,$$

dann ergibt sich

$$\int_{h\pi}^{(h+2)\pi}\frac{1}{(y-\eta)^\beta}e^{c\eta}\cos\eta\,d\eta=(-1)^h e^{ch\pi}\int_0^{2\pi}\frac{1}{(t-\tau)^\beta}e^{c\tau}\cos\tau\,d\tau\,.$$

Das letztere Integral kann beschrieben werden wie folgt:

$$\int_0^{\pi/2}\left[\frac{e^{c\tau}}{(t-\tau)^\beta}-\frac{e^{c(\pi-\tau)}}{(t-\pi+\tau)^\beta}-\frac{e^{c(\pi+\tau)}}{(t-\pi-\tau)^\beta}+\frac{e^{c(2\pi-\tau)}}{(t-2\pi+\tau)^\beta}\right]\cos\tau\,d\tau$$

$$=\int_0^{\pi/2}(Y_1-Y_2-Y_3+Y_4)\cos\tau\,d\tau\ .$$

Nennt man $a_1=t-\tau$, $b_1=t-2\pi+\tau$ und $a_2=t-\pi+\tau$, $b_2=t-\pi-\tau$, so ergeben sich $a_1+b_1=a_2+b_2=2t-2\pi(>0)$ und $a_1-b_1=2\pi-2\tau$, $a_2-b_2=2\tau$, wo $0<2\tau<\pi$ ist, daraus $a_1-b_1>a_2-b_2$ folgt. Deswegen erhält man unter Anwendung von Lemma 1

$$a_1b_1<a_2b_2,\quad \text{d.h.}\quad (t-\tau)(t-2\pi+\tau)<(t-\pi+\tau)(t-\pi-\tau),$$

und weiter

$$\frac{e^{c\tau}\,e^{c(2\pi-\tau)}}{(t-\tau)^\beta(t-2\pi+\tau)^\beta}>\frac{e^{c(\pi-\tau)}\,e^{c(\pi+\ )}}{(t-\pi+\tau)^\beta(t-\pi-\tau)^\beta},\quad\text{d.h.}\quad Y_1Y_4>Y_2Y_3.$$

Anderseits ist Y monoton zunähmend, somit $Y_1<Y_2$ und $Y_3<Y_4$, mithin wird $Y_4-Y_1>Y_3-Y_2$. Daher nach Lemma 2 gilt

$$Y_1+Y_4>Y_2+Y_3 \text{ und folglich } \operatorname{sg}\int_{h\pi}^{(h+2)\pi}=(-1)^h .$$

Sonach hat man $(-v_{2q-1}+v_{2q})>0$, $(v_{2q}-v_{2q+1})<0$ und schließlich $v_{2q-1}<v_{2q}<v_{2q+1}$. Also ist die Folge $\{v_i\}$ monoton wachsend, wie oben erwähnt. Daraus aber ergeben sich für $y=(2q+\varepsilon')\pi$, $0\leqq\epsilon'\leqq1/2$

$$\int_0^y=(v_2-v_1)+(v_4-v_3)+\cdots\cdots+(v_{2q}-v_{2q-1})+v'_{2q+\varepsilon'}>0 ,$$

und für $y=(2q+1+\varepsilon')\pi$, $0\leqq\epsilon'\leqq1/2$

$$\int_0^y=-v_1-(v_3-v_2)-\cdots\cdots-(v_{2q+1}-v_{2q})-v'_{2q+1+\varepsilon'}<0 .$$

Also verschwindet $U(y)$ nur auf dem zweiten und vierten Quadranten des Argumenten y (Modulo 2π).

Könnten wir erweisen, daß $V(y)$ auf dem oben genannten Quadranten keineswegs verschwindet, so würde es bewiesen werden, daß $W=U+iV\neq0$ in dem Gebiet $1>\alpha>0$, $x>\alpha-1$ ist. Zwar in diesem Gebiet kann man nach der vorigen Weise sehen, daß bezw. $V=-$, $+$, $+$ wird, je nachdem $y=2q\pi$, $(2q+1)\pi$ oder $(2q+1/2)\pi$ ist. Da aber das Zeichen von V für $y=(2q+3/2)\pi$ doch unklar ist, so, um den Beweis zu ergänzen, muß ich mich mit etwas anderen Methode behelfen.

Obgleich W eine analytische Funktion von

$$z=x+yi=r(\cos\theta+i\sin\theta)=y(c+i), \qquad c=\cot\theta$$

ist, vermag sie doch als diejenige von y allein gedacht werden, falls θ als fest betrachtet wird. Da wir schon den Falle $\theta=n\pi$ $(y=0)$ ausschlossen haben, so ist $W(z)=W(y,\theta)$ zwar regulär in bezug auf y. Folglich kann der Ausdruck (2) umformt werden zur neuen Gestalt:

$$W=U+iV=\int_0^1\frac{(1-t)^{\alpha-1}}{\Gamma(\alpha)}\exp\{(c+i)yt\}\,dt , \qquad (\alpha>0)$$

worin $c=\cot\theta$ $(\neq\infty)$ bloß als Parameter auftritt, und unabhängig von y ist. Wir gewinnen deshalb

$$\frac{dW}{dy}=\frac{c+i}{\Gamma(\alpha)}\int_0^1 t(1-t)^{\alpha-1}\exp\{(c+i)yt\}\,dt ,$$

was durch partielle Integration

$$\frac{dW}{dy}=\left(c+i-\frac{\alpha}{y}\right)W+\frac{1}{y\Gamma(\alpha)} \tag{3}$$

liefert, und nochmalige Differentiation erteilt

$$\frac{d^2W}{dy^2}=\left[\left(c+i-\frac{\alpha}{y}\right)^2+\frac{\alpha}{y^2}\right]W+\left(c+i-\frac{1+\alpha}{y}\right)\frac{1}{y\Gamma(\alpha)} . \tag{4}$$

Angenommen, nun es geschieht, daß

$$W=U+iV=0, \tag{5}$$

dann müßen vermöge (3) und (4) gelten

$$\frac{dW}{dy}=\frac{dU}{dy}+i\frac{dV}{dy}=\frac{1}{y\Gamma(\alpha)},$$

$$\frac{d^2W}{dy^2}=\frac{d^2U}{dy^2}+i\frac{d^2V}{dy^2}=\left(c-\frac{1+\alpha}{y}\right)\frac{1}{y\Gamma(\alpha)}+\frac{i}{y\Gamma(\alpha)}.$$

Damit lauten für imaginäre Teile

$$V=0,\quad \frac{dV}{dy}=0,\quad \frac{d^2V}{dy^2}=\frac{1}{y\Gamma(\alpha)}\neq 0;$$

und daher muß dort ein Extremum vorhanden sein. Aber bei positivem $\alpha\neq 1$ hat dies keineswegs geschehen, wie wir schon oben so ausführlich gesehen haben. Demnach ist die Annahme (5) zwar unmöglich, was eben zu beweisen ist.

Infolgedessen kann man dartun auf ähnlicher Weise, wie in der Ende des Abschnittes 2, daß der Kurvenzug

$$H^\alpha e^z=U(x,y,\alpha)+i\,V(x,y,\alpha)=0$$

für den parametrige Wert $0<\alpha<1$ durchaus in der Linke der Gerade $x=\alpha-1$ liegen soll. Außerdem vermute ich, daß diese Kurven vielleicht von der Gerade $x=\alpha-1$ asymptotisch im Unendlichen berührt werden.

§ **4.** Bekanntlich verschwindet keine von beiden Funktionen cos z, sin z bis auf reelle Achse, was auch bei etwaiges $\alpha>0$ für Funktionen

$$H^\alpha \cos z=\sum_{n=0}^{\infty}(-1)^n z^{2n}/\Gamma(2n+\alpha+1),$$

sowie

$$H^\alpha \sin z=\sum_{n=1}^{\infty}(-1)^{n-1}\, z^{2n-1}/\Gamma(2n+\alpha)$$

ebenso gut gilt. In der Tat haben beide reelle Funktionen

$$H^\alpha \begin{matrix}\cos\\ \sin\end{matrix} x=\frac{1}{x^\alpha}\int_0^x\frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)}\begin{matrix}\cos\\ \sin\end{matrix}\,\xi\, d\xi$$

für $2>\alpha\geqq 0$ bezw. für $1>\alpha\geqq 0$ je zwei Nullstellen zwischen $2n\pi$ und $2(n+1)\pi$ (jedoch für $\alpha>2$ bezw. $\alpha>1$ gibt es keine Nullstell außer $x=0$), wie man leicht nach den oben benutzten Methode zeigen kann. Also durch H^α-Verfahren bewegen sich die Nullstellen dieser Funktionen auf reeller Achse hin und wieder.

Zum Beispiele verschwindet die Funktion

$$H^1 \cos z=\frac{\sin z}{z}=(\sin x\,\cosh y+i\,\cos x\,\sinh y)/(x+iy)$$

für $x=n\pi\,(n\neq 0)$ $y=0$. Da aber die dortige Determinante

$$D=\begin{vmatrix}\frac{\partial U}{\partial x} & \frac{\partial U}{\partial y}\\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y}\end{vmatrix}=\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2=\frac{1}{n^2\pi^2}\neq 0$$

ist, so muß je eine Kurve in der z-Ebene

$$U(x,y,\alpha)=0,\qquad V(x,y,\alpha)=0$$

durch den Punkt $(x=n\pi, y=0, \alpha=1)$ existieren. Alle diese Punkte liegen durchaus auf reeller Achse. Unsere Verfahren H^α bestimmt also eine homeomorphische Abbildung der Nullstellen von $\cos z$ auf diejenigen von $H^\alpha \cos z$ in der rellen Achse.

Note on Inverses in Rings.

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Let R be an arbitrary ring. Two idempotent elements e and f are called isomorphic in R if there exist two elements a and b such that $ab=e$ and $ba=f$. We write then $e\cong f$. Clearly, by this definition, the set of all idempotent elements in R are divided into classes of idempotent elements. By [1][1], we may assume in the above definition that $a \in eRf$ and $b \in fRe$.

Lemma 1. *If an idempotent element e is isomorphic to zero, then $e=0$.*

Lemma 2. *If two idempotent elements e and f are isomorphic and if e is a sum of two orthogonal idempotent elements e_1 and e_2, then $f=f_1+f_2$, where f_1 and f_2 are orthogonal idempotent elements such that $f_1\cong e_1$ and $f_2\cong e_2$.*

Proof. From $e=ab$ and $f=ba$, we have $be=fb$ and so $bea=fba=f$. Hence $f=bea$ $=be_1a+be_2a$. We see easily that be_1a and be_2a are orthogonal idempotent elements such that $be_1a\cong e_1$ and $be_2a\cong e_2$.

Lemma 3. (Azumaya [1]). *Two idempotent elements e and f are isomorphic if and only if the left ideals Re and Rf are R-isomorphic: $Re\cong Rf$.*

Lemma 4. *If the left ideals Re and Rf are R-isomorphic, then the subrings eRe and fRf are isomorphic.*

Proof. From $e=ab$ and $f=ba$ ($a \in eRf$, $b \in fRe$), we see that eRe and fRf are isomorphic under the mapping $x\longrightarrow bxa$ ($x \in eRe$).

In the following we assume that R contains an identity 1. Generally, $ab=1$ in R does not imply $ba=1$.

Lemma 5. *The following conditions are equivalent:*

(i) $ab=1$ *in* R *implies always* $ba=1$.

(ii) *If* $e\cong 1$, *then* $e=1$.

(iii) *For any idempotent element* $e\neq 1$, R *and* Re *are not* R-*isomorphic.*

Proof. Suppose that R contains a pair of elements a, b such that $ab=1$ but $ba\neq 1$. Then ba is an idempotent element and $ba\cong 1$.

Theorem 1. *If R is a ring with an identity that contains two elements a and b such that $ab=1$, $ba\neq 1$, then*

(i) *R contains an infinite number of idempotent elements e_i such that $e_i\cong 1$.*

(ii) *R contains an infinite number of subrings R_i such that the R_i are isomorphic to R.*

(iii) *R contains a left ideal that is a direct sum of an infinite number of R-isomorphic left ideals* [2].

(iv) *There exists an infinite properly descending chain of principal left ideals generated by idempotent elements:*

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1) Numbers in brackets refer to the references at the end of the paper.

$$R = Re_0 \supset Re_1 \supset Re_2 \supset \cdots\cdots \qquad (e_0 = 1)$$

such that the factor spaces Re_i/Re_{i+1} $(i=0,1,2,\cdots\cdots)$ *are R-isomorphic.*

Proof. Since ba is an idempotent element, we have $1=(1-ba)+ba$, where $1-ba$ and ba are orthogonal idempotent elements. Then we have by Lemma 2 $ba=(ba-b^2a^2)+b^2a^2$, where $ba-b^2a^2$ and b^2a^2 are orthogonal idempotent elements and so

$$1=(1-ba)+(ba-b^2a^2)+b^2a^2 ,$$

where

$$1-ba \cong ba-b^2a^2, \qquad 1 \cong ba \cong b^2a^2 .$$

Continuing in this way, we have for any positive integer n

$$1=\sum_{i=1}^{n}(b^{i-1}a^{i-1}-b^ia^i)+b^na^n \qquad (a^0=b^0=1),$$

where

$$1 \cong ba \cong b^2a^2 \cong \cdots\cdots \cong b^na^n ,$$

$$1-ba \cong ba-b^2a^2 \cong \cdots\cdots \cong b^{n-1}a^{n-1}-b^na^n .$$

Since $1-ba \neq 0$, we see from Lemma 1 that $b^{i-1}a^{i-1}-b^ia^i \neq 0$ and so $b^ia^i \neq b^ja^j$ $(i \neq j)$. Hence there exists an infinite number of idempotent elements b^ia^i such that $b^ia^i \cong 1$. If we set $b^ia^i=e_i$, then

$$R \supset e_1Re_1 \supset e_2Re_2 \supset \cdots\cdots$$

and by Lemma 4

$$R \cong e_iRe_i \qquad (i=1,2,3,\cdots\cdots).$$

Further if we set $b^{i-1}a^{i-1}-b^ia^i=f_i$, then the left ideal $\mathfrak{l}=\sum_i Rf_i$ is a direct sum of R-isomorphic left ideals Rf_i. Finally

$$R=Re_0 \supset Re_1 \supset Re_2 \supset \cdots\cdots$$

is an infinite properly descending chain of left ideals such that $Re_i/Re_{i+1} \cong Rf_1$.

For example, let R be a (complete) direct sum of an infinite number of simple algebras. Then, by Theorem 1 (iv), we see that $ab=1$ in R always implies $ba=1$.

Theorem 2. *Let R be a ring with an identity. Suppose that R splits into a direct sum of an infinite number of left ideals:*

$$R=\mathfrak{l}_1+\mathfrak{l}_2+\mathfrak{l}_3+\cdots\cdots$$

If R contains an infinite number of R-isomorphic left ideals $\mathfrak{l}_i$, *then there exists a pair of elements a, b in R such that* $ab=1$ *but* $ba \neq 1$.

Proof. We see easily that every $\mathfrak{l}_i$ is a left ideal generated by an idempotent element: $\mathfrak{l}_i=Re_i$. Let us assume

$$Re_{i_1} \cong Re_{i_2} \cong Re_{i_3} \cong \cdots\cdots$$

We may assume without loss of generality that $e_{i1}=e_1$. Let $Re_{i_{k-1}}$ be R-isomorphic to Re_{i_k} under the mapping $a_{i_{k-1}} \longrightarrow a'_{i_k}$. Then the mapping

$$a=\sum_{i=1} a_i \longrightarrow a^*=\sum_{i=2} a_i^* \qquad (a_i,\ a_i^* \in Re_i),$$

where $a_i^*=a_i$ if $i \neq i_k$ $(k=1, 2,\cdots\cdots)$ and $a^*_{i_k}=a'_{i_k}$ $(k=2, 3,\cdots\cdots)$, gives the R-isomorphism between R and $R(1-e_1)$. Hence, by Lemma 5, R contains a pair of elements a, b such that $ab=1$ but $ba \neq 1$.

We denote by $r(a)$ [$l(a)$] the right [left] annihilator of an element a.

Lemma 6. *If $ab=1$, then $l(a)=0$ and $r(a)=r(ba)=(1-ba)R$.*

Lemma 7. *If an element a of a ring with an identity has a unique right inverse b, then b is the inverse of a.*

Proof. Since $a(b+1-ba)=1$, we have $1-ba=0$.

Theorem 3 (Jacobson [2]). *If R contains two elements a and b such that $ab=1$ but $ba\neq 1$, then the element a has an infinite number of right inverses.*

Proof. Since $l(a)=0$, we have $a^i\neq 0$ $(i=1, 2,\cdots\cdots)$ and so $(1-ba)a^i\neq 0$. Moreover we have $1-ba\neq(1-ba)a^i$ because $((1-ba)a^i)^2=0$. Hence

$$(1-ba)a^i\neq(1-ba)a^j \qquad (i\neq j).$$

If we set $b_i=b+(1-ba)a^i$, then $ab_i=1$ $(i=0, 1, 2,\cdots\cdots)$[2].

If b and b' are two distinct right inverses of a, then, by Lemma 6, $ba\neq b'a$. But we have

$$Rb'a=Ra=Rba\ .$$

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[1] G. AZUMAYA, On generalized semi-primary rings and Krull-Remak-Schmidt's theorem, Jap. J. Math., Vol. 19 (1948).

[2] N. JACOBSON, Some remarks on one-sided inverses, Proc. Amer. Math. Soc., Vol. 1 (1950).

2) The b_i thus constructed are equal to those in [2], Theorem 3.

Remarks on the Convexity of Connected Sets.

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The set M in the separable real Banach space Ω [1] is said to be locally convex if every point x of the closure $\overline{M}$ is a convex point, that is, there is a positive number δ such that $U_\varepsilon(x)\cap M$[1], if non-null, is convex for any positive $\varepsilon\leqq\delta$. We have proved already in [2] that if M is locally convex and arcweise connected, its interior M^i is cnvex.

In this note we shall introduce the concept of homogeneity into local convex sets to lay more somewhat firmly the foundation of our theory and then shall give the proof of main theorem by discussing the case where M is closed under the weaker condition and some assumption.

1. Preliminaries.

We denote by $S(M)$ the subspace[2] spanned by M i.e., the minimal subspace of Ω which contains M. If $S(U_\varepsilon(x)\cap M) = S(U_{\varepsilon'}(y)\cap M)$for every different $x, y\in\overline{M}$ and every positive numbers ε, ε', then M is said to be homogeneous. This definition is equivalent to the equality that $S(U_\varepsilon(x)\cap M)=S(M)$ for every $x\in M$ and every $\varepsilon>0$.

Lemma 1. *If M is convex, then M is homogeneous.*

Proof. Let $U_\varepsilon(a)$ be a neighbourhood of a point $a\in\overline{M}$.

Suppose $S(U_\varepsilon(a)\cap M)\subsetneqq S(M)$, then there is a point $b\in S(M)-S(U_\varepsilon(a)\cap M)$. It can be easily seen that $f(\lambda)=(1-\lambda)a+\lambda b$, $0<\lambda<\frac{\varepsilon}{\|a-b\|}$, belongs to $S(M)-S(U_\varepsilon(a)\cap M)$. This contradicts to the definition of $S(U_\varepsilon(a)\cap M)$.

More precisely,

Lemma 2. *If M is connected and locally convex, then M is homogeneous.*

Proof. Suppose that the theorem is not true. Let $U_\varepsilon(a)$ be a neighbourhood of a point $a\in\overline{M}$. We denote by X the set of points x belonging to $\overline{M}$ which satisfy $S(U_\delta(x)\cap M)=S(U_\varepsilon(a)\cap M)$ for every δ and fixed ε. Let $Y=\overline{M}-X$. Then $X\neq 0$, and $Y\neq 0$. Since M is connected, $\overline{M}$ is also so, that is, $X'\cap Y\neq 0$[3] or $X\cap Y'\neq 0$, where we take up $X'\cap Y\neq 0$ (similarly in the other case). Let $b\in X'\cap Y$. We can choose a neighbourhood $V(b)$ which contains $x\in X$ and whose intersection with M is convex. By Lemma 1, $V(b)\cap M$ is homogeneous, that is, $S(U_\delta(x)\cap M)=S(U_\zeta(b)\cap M)$; accordingly we have $S(U_\eta(a)\cap M)=S(U_\zeta(b)\cap M)$ for any η, $\zeta>0$, contradicting to the assumption that $b\in Y$. Lemma 2 has thus been proved.

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1) $U_\varepsilon(x)=E[z\,;\,\|z-x\|<\varepsilon]$.
2) Here the subspace means the linear subspace.
3) X' is the derived set of X.

By a relative interior (exterior) point of M, hereafter, we mean the interior (exterior) point[4] of M in the subspace $S(M)$. The set of all relative interior (exterior) points of M is denoted by M^i (M^e) which is called the relative interior (exterior) of M.

Now we suppose that convex sets always contain relative interior points [3]. Then we have readily

Lemma 3. *If M is connected and locally convex, then it holds that $\overline{M}=\overline{M^i}$.*

Proof. Clearly $\overline{M}\supset\overline{M^i}$. We now show $\overline{M}\subset\overline{M^i}$. Taking any $x\in\overline{M}$, $U_\varepsilon(x)\cap M$ is convex for some $\varepsilon>0$. By the presupposition and Lemma 2, there are $x_0\in U_\varepsilon(x)\cap M$ and $\delta>0$ such that

$$S(M)\cap U_\delta(x_0)=S(U_\varepsilon(x)\cap M)\cap U_\delta(x_0)\subset U_\varepsilon(x)\cap M\subset M.$$

Hence we have $\overline{M}\subset\overline{M^i}$.

Further we arrange three lemmas [4].

Lemma 4. *Let a, $b\in\Omega$* and $c=(1-\lambda)a+\lambda b$. *Given a neighbourhood $U(c)$, we can find $V(b)$ such that $[a, x]\cap U(c)\neq 0$[5] for any $x\in V(b)$.*

Lemma 5. *Let M be a convex set, and M^* its bounday. If $a\in M^i$, $r\in M^*$, then $(a, \overrightarrow{r)}\subset M^e$ where $(a, \overrightarrow{r})=[z;\ z=(1-\lambda)a+\lambda r,\ \lambda>1]$*

Lemmn 6. *Let b be a convex point of M and let $[a, b)\subset M^i$. Then there is a suitable neghbourhood $U(b)$ of b such that $[a, x)\subset M^i$ for every $x\in U(b)\cap M$.*

Lemma 7. *If M is connected and locally convex, then $(\overline{M})^i=M^i$.*

Proof. We can show easily that any boundary point of M has relative interior points and relative exterior points of M in its arbitrary neighbourhoods. Therefore we have $(\overline{M})^i\subset M^i$. Clearly $(\overline{M})^i\supset M^i$.

Lemma 8. *The relative interior of a convex set is also convex [5].*

2. Main Theorem.

We shall have

Theorem 1. *If M is connected and locally convex, then its relative interior M^i is convex.*

In order to prove Theorem 1, it is sufficient that we prove Theorem 1′ as follows.

Theorem 1′ *If M is closed connected and locally convex, then M is convex.*

In fact, it is readily seen that the two hold: (1) if M is connected, $\overline{M}$ is connected, (2) if M is locally convex, $\overline{M}$ is locally convex. If $\overline{M}$ is proved to be convex by Theorem 1′, it will be concluded that M^i is convex by Lemma 7 and 8.

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4) Precisely, the point x is called a relative interior point if $S(M)\cap U_\varepsilon(x)\subset M$ for some $\varepsilon>0$, and x a relative exterior point if $S(M)\cap U_\varepsilon(x)\subset CM$ (complement of M).

5) The notations $[a, x]$, $[a, x)$ are defined in [1] p. 26.

Proof of Theorem 1'.

Suppose that M is not convex, then there are points $a, b \epsilon M$ such that $[a, b] \not\subset M$, that is, $[a, b]$ contains a relative exterior point of M, for M is closed. Without restriction the point b may be considered to be a relative interior point of M by Lemma 3 and 4.

Let X be the set composed of all points x of M such that $[b, x] \subset M$ and let Y be $M-X$. Then $M=X \cup Y$, $X \neq O$, $Y \neq O$. Since M is connected, it holds that either $X' \cap Y \neq O$ or $X \cap Y' \neq O$; let, for instance, $r \epsilon X \cap Y'$ (being similar in other case). Then $[b, r] \subset M$ and a neighbourhood $U_{\xi}(r)$ contains a point $p \epsilon Y$ yielding $[b, p] \not\subset M$, while $U_{\xi}(r)$ can be chosen so that $U_{\xi}(r) \cap M$ is coevex. Now every point of $[r, p]$ included by M is represented as $f(\lambda)=(1-\lambda)p+\lambda r$, $0 \leqq \lambda \leqq 1$.

Letting $\lambda_0 = sup[\lambda \,|\, [b, f(\mu)] \not\subset M \text{ for every } \mu,\ 0<\mu<\lambda \leqq 1$, it follows that a boundary point of M lies in $(b, f(\lambda_0))$. Because if it held that $[b, f(\lambda_0)) \subset M^i$, there would exist some neighbourhood $U_\delta(f(\lambda_0))$ such that $[b, x) \subset M^i$ for any $x \epsilon U_\delta(f(\lambda_0)) \cap M$ by Lemma 6. Consequently it would follow that $[b, f(\mu)] \subset M$ whenever $\lambda_0 - \varepsilon < \mu < \lambda_0$ for an adequate $\varepsilon > 0$. This contradicts with the assumption of λ_0.

Let $g(\nu)=(1-\nu)b+\nu f(\lambda_0)$ be every point of $[b, f(\lambda_0)]$ and $\nu_0 = inf[\nu \,|\, g(\nu) \epsilon M^*]$. Then $g(\nu_0)$ is clearly a boundary point and $V \cap M$ is convex for an adequate neighbourhood V of $g(\nu_0)$. There is $\eta > 0$ such that $g(\nu) \epsilon V$ for every ν, $\nu_0 - \eta < \nu < \nu_0 + \eta$; in detail, the points $g(\nu')$ are interior points of $M \cap V$ for ν', $\nu_0 - \eta < \nu' < \upsilon_0$ and $g(\nu'')$ are exterior points of $M \cap V$ for ν'', $\nu_0 < \nu'' < \nu_0 + \eta$ by Lemma 5, accordingly $g(\nu'')$ has its neighbourhood $W \subset M'$. Utilizing Lemma 4, it holds that $[b, y] \cap W \neq O$ for every $y \epsilon N(f(\lambda_0);\delta)$ where $N(f(\lambda_0);\delta)$ is some neighbourhood of $f(\lambda_0)$. Especially for some ζ_0, we have $[b,\ f(\lambda_0+\delta)] \cap W \neq O$, $0<\zeta<\zeta'$, contradicting with the definition of λ_0. Thus the proof of the theorem has been completed.

At the end I wish to express my hearty thanks to Prof. H. Terasaka of Osaka University for his kind guidance.

Notes.

〔1〕 Separability of Ω exerts an effect on Lemma 6. With respect to this, see

T. TAMURA, *On a relation between local convexity and entire convexity*, Journal of Science of the Gakugei Faculty, Tokushima University, Vol. 1, (1950) p. 30 and p. 27.

〔2〕 The paper 〔1〕, pp. 25—30.

〔3〕 This supposition is serious, but we don't here treat it in detail. Of course, it holds naturally when Ω is finite dimensional.

〔4〕 Lemma 4, 5, and 6 equal to Corollary 1 (p. 26), Corollary 2 (p. 27), and Lemma 3 (p. 27) respectively of this Journal Vol. 1 (1950).

〔5〕 This is proved easily by the preceeding lemmas.

Addendum.

I would like to express my heartfelt gratitude to Prof. M. M. Day (Urbana, I 11.) for his kind remarks in the Mathematical Reviews, Vol. 13, No. 5, as to my paper of this Journal Vol. 1.

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On the Regularization of Metrics in Complex Space.

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Introduction

In the preceding papers,[1] we have discussed some properties on geometry in complex space, but have not yet considered those metric properties. In complex space, there are Unitary metrics and Hermite-Kähler metrics[2] corresponding to Euclidean metrics and Riemannian metrics in real space, respectively. But as these metric functions are not regular, so it is not convenient to introduce the theories of regular functions into the studies of geometry in complex space. So it is requested to obtain such regular functions as whose norms have some geometrical meanings in complex space; and to do so we shall call "Regularization of Metrics".

In this paper we shall give some regular functions with respect to the regularization of metrics in two dimentional complex space. These results are expected to be interest in the theories of geometry in complex space and the theories of functions of several complex variables.

§ 1. Distance between two points on a complex straight line.

To consider the distance between two points on a complex straight line, we can put the two given points to O(o, o) and P(z^1, z^2) without loss of generality. As z^k means x^k+iy^k (k=1, 2) and the metrics obey Unitary metrics, S=OP is given by

$$S=\sqrt{{x^1}^2+{y^1}^2+{x^2}^2+{y^2}^2} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1).$$

So far as S is given by (1) we know nothing that S means what kind of regular functions. Then we shall give a regular function S, whose norm is given by (1) and whose geometrical meanings are given in complex space. In special case, if P is on the holomorphic plane $z^1=0$,[3] S is given by x^2+iy^2, and $\|S\|=\sqrt{{x^2}^2+{y^2}^2}$, so S is expressible by a regular function of x^2 and y^2. So in this case we can say that our assertion is attained. Then at first, we shall transform the equation of the given holomorphic plane to the form $z^1=O$ by some suitable congruent transformations. On these transformations we have explained in the preceding paper[4] already.

So after the transformation, the new variable z^2 means the requested distance S. By way of prevention against confution, we take X^k, Y^k as new variables, instead of the old variables x^k, y^k.

If we take $P_0(z_0^1, z_0^1)$ to any point on the holomorphic plane OP, then the equations of the holomorphic plane OP are

$$\text{(I)}\quad \begin{aligned} &\frac{x_0^2}{S_0}x^1-\frac{y_0^2}{S_0}y^1-\frac{x_0^1}{S_0}x^2+\frac{y_0^1}{S_0}y^2=0\\ &\frac{y_0^2}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2=0 \end{aligned}\qquad \cdots\cdots\cdots(2).$$

In (2) S_0 means $\sqrt{x_0^{2^2}+y_0^{2^2}+x_0^{2^2}+y_0^{2^2}}$

If we transform $x^k y^k$ under the adjoint transformation of (I), *i, e,*

$$\begin{pmatrix} \frac{x_0^2}{S_0} & \frac{y_0^2}{S_0} & \frac{x_0^1}{S_0} & -\frac{y_0^1}{S_0}\\ -\frac{y_0^2}{S_0} & \frac{x_0^2}{S_0} & \frac{y_0^1}{S_0} & \frac{x_0^1}{S_0}\\ -\frac{x_0^1}{S_0} & -\frac{x_0^2}{S_0} & \frac{x_0^2}{S_0} & -\frac{y_0^2}{S_0}\\ \frac{y_0^1}{S_0} & -\frac{x_0^1}{S_0} & \frac{y_0}{S_0} & \frac{x_0^2}{S_0} \end{pmatrix}\qquad\cdots\cdots\cdots(3)$$

to X^kY^k, we have the follwings,

$$\begin{aligned} X^1&=\frac{x_0^2}{S_0}x^1-\frac{y_0^2}{S_0}y^1-\frac{x_0^1}{S_0}x^2+\frac{y_0^1}{S_0}y^2\\ Y^1&=\frac{y_0^2}{S_0}x^1+\frac{x_0^2}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2\\ X^2&=\frac{x_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1+\frac{x_0^1}{S_0}x^2+\frac{y_0^2}{S_0}y^2\\ Y^2&=-\frac{y_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2 \end{aligned}\qquad\cdots\cdots\cdots(4).$$

In (4) we can put X^1, Y^1 to 0, for $P(z^1, z^2)$ is on the plane (I), so we get

$$S=X^2+i\,Y^2=\left(\frac{x_0^1}{S_0}x^1+\frac{y_0^1}{S_0}y^1+\frac{x_0^2}{S_0}x^2+\frac{y_0^2}{S_0}y^2\right)+i\left(-\frac{y^{01}}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2\right)\cdots(5).$$

We can ascertain that $\|S\|$ is $\sqrt{x^{1^2}+y^{1^2}+x^{2^2}+y^{2^2}}$ for the because of $X^1=Y^1=O$.

Obviously S is a regular function of z^1, z^2, so our assertion is attained.

Then we get the following theorem.

Theorem 1. *The distance of two point on a complex straight line is given by the positions of the points only, and is independent on their passing through arcs on the holomorphic plane.*

Proof. The distance of given two points P, Q along an arc $\widehat{PQ}$ is given by $\int_{\widehat{PQ}} ds$, and ds is a regular function of z^k, so the integral is independent on the choice of arcs. Then we can say that the arc length along a closed curve is zero on a holomorphic plane.

In the Theorem 1 we can see the strong one dimentional properties of holomorphic planes, inspite of the two dimentional properties of general planes.

Though S is given by (5) as a regular function of z^k, but it is not very simple. When the equations of a holomorphic plane are given by their parametric forms, S is reduced to more simple forms.

The parametric equations of the holomorphic plane (I) is given by where

$z^k=(P_k+iQ_k)(u+iv)$ ···········(6) ($k=1, 2$)

u, v are the parameters. So P_o is on the plane (I) we get

$$z_o^k=(P_k+iQ_k)(u_o+iv_o) \cdots\cdots (7)$$

We have from (6) and (7)

$$x^k=P_ku-Q_kv \quad y^k=Q_ku+P_kv \quad x_o^k=P_ku_o-Q_kv_o \quad y_o^k=Q_ku_o+P_kv_o \cdots\cdots (8).$$

If we substitute these x^ky^k, $x_o^ky_o^k$ to (5) we get $S=\sqrt{g}e^{i\varphi}(u+iv)$ ·············(9)

In (9) $\sqrt{g}$ means $\sqrt{[PP]+[QQ]}$ where $[PP]=\sum_{K=1}^{2}P_k^2$, $[QQ]=\sum_{K=1}^{2}Q_k^2$, and $e^{i\varphi}$ is an indeterminate phase factor whose norm is unity. In (9) P, Q are constants, so S is a regular functions of u and v.

§ 2. Trigonometric functions, Triangular area.

On the intersecting angle between two given complex straight line, we have explained already, but we shall explain here systematically again.

Let us put the coordinates of P, P′ to (z^1, z^2), (z'^1, z'^2), and the angle between two complex lines OP and OP′ to θ.

The we can define that

$$\cos\theta=\sum_{K=1}^{2}\frac{z^k}{S}\,\frac{z'^k}{S'} \cdots\cdots (10)$$

where, $\overline{OP}=S$ $\overline{OP}\doteq S'$. If

we put the values of z^k, z'^k in (6) to (10), so the parameters u,u', v,v' are eliminated, then we get

$$\cos\theta=\frac{\{[PP']+[QQ']\}+i\{[PQ']-[P'Q]\}}{\sqrt{g}\,\sqrt{g'}}\,e^{i} \cdots\cdots (11).$$

In the equation (11), $[PP']$, $[QQ']$ are $\sum_{K=1}^{2}P_kP_k'$, $\sum_{K=1}^{2}Q_kQ_k'$ respectively. From (11)

we get $$\|\cos\theta\|=\sqrt{\frac{\{[PP']+[QQ']^2+\{[PQ']-[Q'Q]\}^2}{gg'}} \cdots\cdots (12).$$

The equation (12) means that $\|\cos\theta\|$ is coincident perfectly to (7) in the preceding paper.[5)]

From the facts of the above, we get an expression of $\cos\theta$, which is a regular function of parameters, and whose norm has geometric meanings in Unitary space. So the regularization of $\cos\theta$ was accomplished completely.

Then we consider the regularization of triangular areas. Let us consider the triangle OPP′, then we can define its area to the form

$$F=\tfrac{1}{2}\{z^1z'^2-z'^1z^2\} \cdots\cdots (13),$$

from an extention of real space. The equation (13) means that F is a regular function of z^k, z'^k, so if the norm of F has geometrical meanings, we can say that the regularization of area was accomplished.

To ascertain that, we may see the relation

$$2\|F\|=\|S\|S'\|\|\sin\theta. \qquad \cdots\cdots(14).$$

To obtain the relation (14), at first, we get from (1) the relation

$\|S\|=\sqrt{x^{1^2}+y^{1^2}+x^{2^2}+y^{2^2}}\ \|S'\|=\sqrt{x'^{1^2}+y'^{1^2}+x'^{2^2}+y'^{2^2}}$, then by substitution of

$\|\cos\theta\|^2=\dfrac{\{[xx']+[yy]\}^2+\{[xy']-[x'y]\}^2}{S^2S'^2}$, which is equivalent to (12), to the relation $\|\sin\theta\|=\sqrt{1-\|\cos\theta\|^2}$, we get

$$\|\sin\theta\|^2=\frac{S^2S'^2-\{[xx']+[yy']\}^2+\{[xy']-[x'y]\}^2}{S^2S'^2}.$$

Then if we compare the value of $\|F\|$, which is reduced from (14), to that of reduced from (13). we can ascertain that these two values of $\|F\|$ are coincident completely. So if we define the triangular area OPP′ by (13), we may accomplish the regularization of triangular area. It is to be noticed that the angle which was used here, is not meant the intersecting angle of the vector $\overline{OP}$ and $\overline{OP'}$, but the intersecting angle of the holomorphic plane OP and OP′. Then if three points O, P, P′, are on a same holomorphic plane, the values of $\sin\theta$ is zero, so it is seen that the triangular area is also zero.

Let us rewriten the form of F by the use of parameters. From (13) we get

$$2F=\left\{\begin{vmatrix}x^1x^2\\x^1x^2\end{vmatrix}-\begin{vmatrix}y^1y^2\\y'^1y'^2\end{vmatrix}\right\}+i\left\{\begin{vmatrix}x^1y^2\\x'^1y'^2\end{vmatrix}+\begin{vmatrix}y^1x^1\\y'^1x'^2\end{vmatrix}\right\}. \qquad \cdots\cdots(15).$$

If we put the values of $x^k y^k$ of (6) to (15), we obtain

$$2F=\begin{vmatrix}(P_1+iQ_1)(u+iv) & (P_2+iQ_2)(u+iv)\\(P_1'+iQ_1')(u'+iv') & (P_2'+iQ_2')(u'+iv')\end{vmatrix}$$
$$=(u+iv)(u'+iv')\left\{\left(\begin{vmatrix}P_1\ P_2\\P_1'P_2'\end{vmatrix}-\begin{vmatrix}Q_1\ Q_2\\Q_1'Q_2'\end{vmatrix}\right)+i\left(\begin{vmatrix}P_1\ Q_2\\P_1'Q_2'\end{vmatrix}+\begin{vmatrix}Q_1\ P_2\\Q_1'Q_2'\end{vmatrix}\right)\right\}. \quad \cdots\cdots(16).$$

In (16) we see that F is a regular function of u, v, and u′, v′. At the end of the section we get the regularization of $\sin\theta$ from the relation $\sin\theta=\dfrac{2F}{SS'}$; the meanings of the norm was explained already. So if we substitute the values of F, S, S' from (9) and (16) to the above we get

$$\sin\theta=\frac{\left(\begin{vmatrix}P_1\ P_2\\P_1'P_2'\end{vmatrix}-\begin{vmatrix}Q_1\ Q_2\\Q_1'P_2'\end{vmatrix}\right)+i\left(\begin{vmatrix}P_1\ P_2\\P_1'P_2'\end{vmatrix}+\begin{vmatrix}Q_1\ P_2\\Q_1'P_2'\end{vmatrix}\right)}{\sqrt{g}\ \sqrt{g'}} \qquad \cdots\cdots(17).$$

From (17) we may say that the regularization of $\sin\theta$ was accomplished. From these relations we can say that the regularization of some fundamental magnitudes in complex space were accomplished. So we shall consider the regularization of arc length of complex curves in next step.

§ 3. Arc length along a complex curve.

The arc length along a complex cnrve is considered as a limiting case of a straight line. From (9) we get $S=\sqrt{g}e^{i\varphi}(u+iv)$ along a complex straight line. In this case $\sqrt{g}$ is constant, so S is a regular function of u and v, but in general case $\sqrt{g}$ is a function of u and v. So if we find a regular function $U(uv)+iV(uv)$ whose norm is $\sqrt{g}\,(u^2+v^2)$, so our regularization is accomplished; but it is impossible in general case. So the regularization of arc length along a complex curve must be considered as a limiting case of a complex straight line. We shall reduce the arc length along a complex curve from the metric properties of a holomorphic surface.

Let P, Q be infinitesimally consecutive points on a holomorphic surface. We may put the arc length $\widehat{PQ}$ to ds, and we get the length of a finite arc, by the integration of ds on the holomorphic surface.

To do that, we consider two tangential planes at P and Q respectively; for these tangential planes are holomorphic, their intersecting point is decided uniquely. So, we put the intersecting point to I, and define the infinitesimal arc length $\widehat{PQ}$, as the sum of the infinitesimal distances $\overline{PI}$ and $\overline{IQ}$ on these tangential holomorphic planes. We do not know whether the sum of these infinitesimal distance are regulsr or not. To these concerning facts we get the following theorem.

Theorem 2. *The infinitesimal arc length along a complex curve is given by a regular function of parameters. So the arc length along a infinitesimally closed curve on a holomorphic surface considered to be zero in locally sense.*

To prove the theorem we put the parameters u, v to zero at the given point O on the holomorphic surface. So the equation of the holomorphic surface in the neighbourhood of O are given by

$$z^k=(P_k+iQ_k)(u+iv)+\frac{1}{2}(L_k-iM_k)(u+iv)^2 \qquad \cdots\cdots\cdots\cdots(18).$$

In (18) P_k, Q_k, Γ_k, M_k are that

$$\frac{\partial x^k}{\partial u}=\frac{\partial y^k}{\partial v}=P_k \quad \frac{\partial y^k}{\partial u}=-\frac{\partial x^k}{\partial v}=Q_k \qquad \cdots\cdots\cdots\cdots(19)$$

$$\frac{\partial^2 x^k}{\partial u_2}=\frac{\partial^2 y^k}{\partial u\partial v}=-\frac{\partial^2 x^k}{\partial v^2}=L_k \quad -\frac{\partial^2 y^k}{\partial u^2}=\frac{\partial^2 x^k}{\partial u\partial v}=\frac{\partial^2 y^k}{\partial v^2}=M_k \quad (k=1.2)$$

in (19) $x^k y^k$ are the real and imaginary part of z^k.

Then the equation (18) are reduced to the form (6)

$$x^k=P_k u-Q_k v+\frac{1}{2}L_k(u^2-v^2)+M_k uv \qquad \cdots\cdots\cdots\cdots(20).$$

$$y^k=Q_k u+P_k v-\frac{1}{2}M_k(u^2-v^2)+L_k uv$$

To prove the Theorem 2 on the holomorphic surface (20), we shall prove the the next lemma.

Lemma. *If we put δs to the infinitesimal arc length $\widehat{OA}$ on the holomorphic surface,* *and ds to the infinitesimal distance $\overline{OA}$ on the holomorphic plane which pass through the two given points O and A, so there are the relation*

$$\delta s=\left(1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta v\right)ds\ , \qquad \cdots\cdots(21),$$

where $\left\{\begin{matrix}1\\11\end{matrix}\right\}$ and $\left\{\begin{matrix}1\\12\end{matrix}\right\}$ are the Christoffel's symbols on the holomorphic surface, and δu, and δv are the differences of the parameters.

To prove the lemma, we shall calculate the arc length $\overset{\frown}{OA}$ by the definition of the above. That is to say, the arc length between O(o, o) and A(δu, δv) on the holomorphic surface is given by $\overline{OI}+\overline{IA}$, where I is the intersecting point of the two tangential planes at O and A. So we shall obtain, at first, the equations of these tangential planes.

In general, the equations of the tangential plane of the surface $x^k=x^k(uv)$ $y^k=y^k(uv)$ $(k=1.2)$ at the point $(u_0\ v_0)$ are given by

$$x^k=x^k(u_0\,v_0)+\left(\frac{\partial x^k}{\partial u}\right)_{\substack{u=u_0\\v=v_0}}u+\left(\frac{\partial x^k}{\partial v}\right)_{\substack{u=u_0\\v=v_0}}v,\ \ y^k=y^k(u_0v_0)+\left(\frac{\partial y^k}{\partial u}\right)_{\substack{u=u_0\\v=v_0}}u+\left(\frac{\partial y^k}{\partial v}\right)_{\substack{u=u_0\\v=v_0}}v \quad\cdots(22),$$

where u, v are the parameters on the tangential plane. So from (19) we see that the equations of tangential planes at O is

(I) $x^k=P_ku-Q_kv \quad y^k=Q_ku+P_kv$ $\cdots\cdots(23)$

and the equations of tangential plane at A is

$$\text{(II)}\quad x^k=P_k\delta u-Q_k\delta v+\frac{1}{2}L_k(\delta u^2-\delta v^2)+M_k\delta u\delta v+(P_k+L_k\delta u+M_k\delta v)\bar{u}$$
$$-(Q_k-M_k\delta u+L_k\delta v)\bar{v} \qquad\cdots\cdots(24).$$
$$y^k=Q_k\delta u+P_k\delta v-\frac{1}{2}M_k(\delta u^2-\delta v^2)+L_k\delta u\delta v+(Q_k-M_k\delta u+L_k\delta v)\bar{u}$$
$$+(P_k+L_k\delta u+M_k\delta v)\bar{v}$$

in (24) $\bar{u}$, $\bar{v}$ are the parameters on the plane (11).

To obtain the coordinates of the intersecting point I, we shall get the values of parameters u, v and $\bar{u}$, $\bar{v}$, which give the same values of x^k, y^k from the equations of (I) and (II), These values of parameters evidently are

$$u=\frac{\delta u}{2}\quad v=\frac{\delta v}{2}\ ,\quad \bar{u}=-\frac{\delta u}{2}\ ,\quad \bar{v}=-\frac{\delta v}{2}\ . \qquad\cdots\cdots(25).$$

From the equations of (I), (II) and (25) we get

$$\overline{OI}=\sqrt{g}\left(\frac{\delta u}{2}+i\frac{\delta v}{2}\right). \qquad\cdots\cdots(26).$$

To obtain $\overline{IA}$ after the rejection of higher terms of δu, δv in

$$\sqrt{\sum(P_k+L_k\delta u+M_k\delta v)^2+\sum(Q_k-M_k\delta u+L_k\delta v)^2},$$

we get

$$\overline{IA}=\sqrt{g}\left\{1+\frac{[PL]-[QM]}{g}\delta u+\frac{[PM]+[QL]}{g}\delta v\right\},$$

then we get

$$\widehat{OA}=\overline{OI}+\overline{IA}$$

$$=\sqrt{g}(\delta u+i\,\delta v)+\sqrt{g}\left\{\frac{[PL]-[QM]}{g}\delta u+\frac{[PM]+[QL]}{g}\delta v\right\}\left(\frac{\delta u}{2}+i\frac{\delta v}{2}\right). \quad\cdots\cdots\cdots\cdots(27).$$

On the other hand we see that the fundamental tensors of the holomorphic surfaces (20) is

$$g^{11}=g^{22}=\frac{1}{g} \quad g_{11}=g_{22}=g \quad\cdots\cdots\cdots\cdots(28)$$

then we get

$$\frac{\partial g_{11}}{\partial u}=\frac{\partial g_{22}}{\partial u}=2\{[PL]-[QM]\}, \quad \frac{\partial g_{11}}{\partial v}=\frac{\partial g_{22}}{\partial v}=2\{[PM]+[QL]\} \quad\cdots\cdots\cdots(29).$$

From (29) we get

$$\left\{\begin{matrix}1\\11\end{matrix}\right\}=\frac{[PL]-[QM]}{g} \qquad \left\{\begin{matrix}1\\12\end{matrix}\right\}=\frac{[PM]+[QL]}{g} \quad\cdots\cdots\cdots\cdots(30).$$

If we put $\widehat{OA}=\overline{OI}+\overline{IA}=\delta s$, $\sqrt{g}\,(\delta u+i\delta v)=ds$, and substitute the values of (30) to the equation (27), we see that the equation (21) is $\delta s=\left(1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta v\right)$.

So the lemma is proved.

From the above lemma, we can prove the Theorem 2 easily. Let us take on the holomorphic surface (20), the three infinitesimally consecutive points O(o, o) $A(\delta u, \delta v)$ B $(\delta u', \delta v')$, then we show that the arc length of closed curve OABO is a second ordered infinitesimal with respect to δu and δv.

If we put the arc length $\widehat{OA}$ $\widehat{AB}$ $\widehat{BO}$ to δs_1, δs_2, δs_3 respectively, we get similarly to (21), the followings

$$\delta s_1=\left\{1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta v\right\}ds$$

$$\delta s_2=-\left\{1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}(\delta u+\delta' u)+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}(\delta v+\delta' v)\right\}ds$$
$$+\left\{1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}(\delta u+\delta' u)\frac{1}{2}\left\{\begin{matrix}I\\12\end{matrix}\right\}(\delta v+\delta' v)\right\}ds' \quad\cdots\cdots\cdots\cdots(31),$$

$$\delta s_3=-\left\{1+\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta' u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta' v\right\}ds'$$

in the equation (31) ds and ds' are given by

$$ds=\sqrt{g}(\delta u+i\,\delta v), \quad ds'=\sqrt{g}(\delta' u+i\,\delta' v). \quad\cdots\cdots\cdots\cdots(32)$$

From (31) evidently we have

$$\delta s_1+\delta s_2+\delta s_3=\left\{\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta v\right\}ds'$$
$$-\left\{\frac{1}{2}\left\{\begin{matrix}1\\11\end{matrix}\right\}\delta' u+\frac{1}{2}\left\{\begin{matrix}1\\12\end{matrix}\right\}\delta' v\right\}ds.$$

If we put the values of ds′ in (32) to the above we get

$$\delta s_1+\delta s_2+\delta s_3=\frac{\sqrt{g}}{2}\left(\begin{Bmatrix}1\\12\end{Bmatrix}-i\begin{Bmatrix}1\\11\end{Bmatrix}\right)(\delta' u\delta v-\delta u\delta' v) \qquad \cdots\cdots(33),$$

From the results of (33) Theorem 2 is proved.

The results of the above seems very likely to the relation of Euclidean space and its connected space. That is in Euclidean space, the direction of a vector returns back to that of the initial position, after the parallel displacement along a finife closed curve; but in the connected space it returns back in the sense of local properties, and it does not, after a finite displacement. So we can say that the theories of regular functions on Gauss's plane correspond to the theories of gemometry in Euclidean space, and the theories of regular functions on the holomorphic surfaces correspond to the theories of geometry in its connected space.

§ 4. Curvature of a complex curve.

We can define the curvature of a complex curve at O, thus.

$$\frac{1}{\rho}=\lim_{A\to 0}\frac{\delta\theta}{\delta s}. \qquad \cdots\cdots(34),$$

In (34) O(o, o) and A $(\delta u, \delta s)$ are two given consecutive points, and δs is the arc length of $\widehat{OA}$, and $\delta\theta$ is the intersecting angle of the tangential planes at O and A respectively. Because of $\delta\theta$ is infinitesimal we can substitute the value of $\sin\delta\theta$ to $\delta\theta$ in (34). From (23) (24) (17) we get

$$\sin\delta\theta=\frac{1}{g}\{(a\delta u-b\delta v)+i\,(b\delta u+a\delta v)\} \qquad \cdots\cdots(35),$$

where a, b are that

$a=P_1L_2-P_2L_1+Q_1M_2-Q_2M_1 \quad b=-P_1M_2+P_2M_1-Q_2L_1+Q_1L_2$ respectively.

Because of the facts that δs is infinitesimal, we can put the value of $\delta s=\sqrt{g}(\delta u+i\delta v)$ to δs in (34), so we get

$$\frac{1}{\rho}=\frac{(a\delta u-b\delta v)+i\,(b\delta u+a\delta v)}{g^{3/2}\,(\delta u+i\,\delta v)}=\frac{(a+ib)}{g^{3/2}} \qquad \cdots\cdots(36).$$

So if the geometrical meanings of the norm of $\frac{1}{\rho}$ are given, from (36) the regularization of $\frac{1}{\rho}$ was accomplished. Concerning the facts, we see that the next theorem is hold.

Theorem 3. *The norm of curvature of a complex curve at any giaon point, is equal to* $\sqrt{-\frac{1}{2}K}$, *where K is the Gauss's total curvature of the holomorphic surface at the given corresponding point.*

Proof. If we construct the Riemann Christoffel's tensor on the holomorphic surface (20), from (28) and (29) we get

$$R^1_{212} = \frac{2\{[LL]+[MM]\}\{[PP]+[QQ]\}-2\{[PL]-[QM]\}^2-2\{[PM]+QL]\}^2}{g^2} \quad \cdots(37).$$

Simplifying the numerator of the above we can reduce it to the form $2(a^2+b^2)$ On the other hand the Gauss's total curvature K is redused to

$$K = -\frac{R_{1212}}{g_{11}g_{22}} = -\frac{g_{11}R^1_{212}}{g_{11}g_{22}} = -\frac{R^1_{212}}{g_{22}} = \frac{-2(a^2+b^2)}{g^3} \quad \cdots\cdots\cdots\cdots\cdots(38).$$

Comparing (38) to (36) we get $\left\|\frac{1}{\rho}\right\| = \sqrt{-\frac{1}{2}K}$ $\cdots\cdots\cdots\cdots\cdots\cdots(39),$

then the theorem was proved.

It is to be noticed that the Gauss's total curvatue of holomorphic surface is a negative quantiiy-

Conclusions.

In this paper we have discussed the metric properties of two dimentional complex space only, then we expect that we refer to the metric properties of higher dimentional complex space in the next paper.

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