On the Regularization of Metrics in Complex Space.

Takaharu MARUYAMA.

Mathematical Institute, Faculty of Engineering, Tokushima University.

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Introduction

In the preceding papers,[1] we have discussed some properties on geometry in complex space, but have not yet considered those metric properties. In complex space, there are Unitary metrics and Hermite-Kähler metrics[2] corresponding to Euclidean metrics and Riemannian metrics in real space, respectively. But as these metric functions are not regular, so it is not convenient to introduce the theories of regular functions into the studies of geometry in complex space. So it is requested to obtain such regular functions as whose norms have some geometrical meanings in complex space; and to do so we shall call "Regularization of Metrics".

In this paper we shall give some regular functions with respect to the regularization of metrics in two dimentional complex space. These results are expected to be interest in the theories of geometry in complex space and the theories of functions of several complex variables.

§ 1. Distance between two points on a complex straight line.

To consider the distance between two points on a complex straight line, we can put the two given points to O(o, o) and P(z^1, z^2) without loss of generality. As z^k means x^k+iy^k (k=1, 2) and the metrics obey Unitary metrics, S=OP is given by

$$S=\sqrt{{x^1}^2+{y^1}^2+{x^2}^2+{y^2}^2} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1).$$

So far as S is given by (1) we know nothing that S means what kind of regular functions. Then we shall give a regular function S, whose norm is given by (1) and whose geometrical meanings are given in complex space. In special case, if P is on the holomorphic plane $z^1=0$,[3] S is given by x^2+iy^2, and $\|S\|=\sqrt{{x^2}^2+{y^2}^2}$, so S is expressible by a regular function of x^2 and y^2. So in this case we can say that our assertion is attained. Then at first, we shall transform the equation of the given holomorphic plane to the form $z^1=O$ by some suitable congruent transformations. On these transformations we have explained in the preceding paper[4] already.

So after the transformation, the new variable z^2 means the requested distance S. By way of prevention against confution, we take X^k, Y^k as new variables, instead of the old variables x^k, y^k.

If we take $P_0(z_0^1, z_0^1)$ to any point on the holomorphic plane OP, then the equations of the holomorphic plane OP are

$$(\mathrm{I})\qquad \begin{aligned} &\frac{x_0^2}{S_0}x^1-\frac{y_0^2}{S_0}y^1-\frac{x_0^1}{S_0}x^2+\frac{y_0^1}{S_0}y^2=0 \\ &\frac{y_0^2}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2=0 \end{aligned} \qquad\cdots\cdots(2).$$

In (2) S_0 means $\sqrt{x_0^{2^2}+y_0^{2^2}+x_0^{2^2}+y_0^{2^2}}$

If we transform $x^k y^k$ under the adjoint transformation of (I), *i, e,*

$$\begin{pmatrix} \frac{x_0^2}{S_0} & \frac{y_0^2}{S_0} & \frac{x_0^1}{S_0} & -\frac{y_0^1}{S_0} \\ -\frac{y_0^2}{S_0} & \frac{x_0^2}{S_0} & \frac{y_0^1}{S_0} & \frac{x_0^1}{S_0} \\ -\frac{x_0^1}{S_0} & -\frac{x_0^2}{S_0} & \frac{x_0^2}{S_0} & -\frac{y_0^2}{S_0} \\ \frac{y_0^1}{S_0} & -\frac{x_0^1}{S_0} & \frac{y_0}{S_0} & \frac{x_0^2}{S_0} \end{pmatrix} \qquad\cdots\cdots(3)$$

to X^kY^k, we have the follwings,

$$\begin{aligned} X^1&=\frac{x_0^2}{S_0}x^1-\frac{y_0^2}{S_0}y^1-\frac{x_0^1}{S_0}x^2+\frac{y_0^1}{S_0}y^2 \\ Y^1&=\frac{y_0^2}{S_0}x^1+\frac{x_0^2}{S_0}y^1-\frac{y_0^1}{S_0}x^2-\frac{x_0^1}{S_0}y^2 \\ X^2&=\frac{x_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1+\frac{x_0^1}{S_0}x^2+\frac{y_0^2}{S_0}y^2 \\ Y^2&=-\frac{y_0^1}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2 \end{aligned} \qquad\cdots\cdots(4).$$

In (4) we can put X^1, Y^1 to 0, for $\mathrm{P}(z^1, z^2)$ is on the plane (I), so we get

$$S=X^2+i\,Y^2=\left(\frac{x_0^1}{S_0}x^1+\frac{y_0^1}{S_0}y^1+\frac{x_0^2}{S_0}x^2+\frac{y_0^2}{S_0}y^2\right)+i\left(-\frac{y^{01}}{S_0}x^1+\frac{x_0^1}{S_0}y^1-\frac{y_0^1}{S_0}x^2+\frac{x_0^2}{S_0}y^2\right)\cdots(5).$$

We can ascertain that $\|S\|$ is $\sqrt{x^{1^2}+y^{1^2}+x^{2^2}+y^{2^2}}$ for the because of $X^1=Y^1=O$.

Obviously S is a regular function of z^1, z^2, so our assertion is attained.

Then we get the following theorem.

Theorem 1. *The distance of two point on a complex straight line is given by the positions of the points only, and is independent on their passing through arcs on the holomorphic plane.*

Proof. The distance of given two points P, Q along an arc $\widehat{\mathrm{PQ}}$ is given by $\int_{\widehat{PQ}} ds$, and ds is a regular function of z^k, so the integral is independent on the choice of arcs. Then we can say that the arc length along a closed curve is zero on a holomorphic plane.

In the Theorem 1 we can see the strong one dimentional properties of holomorphic planes, inspite of the two dimentional properties of general planes.

Though S is given by (5) as a regular function of z^k, but it is not very simple. When the equations of a holomorphic plane are given by their parametric forms, S is reduced to more simple forms.

The parametric equations of the holomorphic plane (I) is given by where

$z^k=(P_k+iQ_k)(u+iv)$ ···········(6) ($k=1, 2$)

u, v are the parameters. So P_o is on the plane (I) we get

$$z_o{}^k=(P_k+iQ_k)(u_o+iv_o) \cdots\cdots (7)$$

We have from (6) and (7)

$$x^k=P_ku-Q_kv \quad y^k=Q_ku+P_kv \quad x_o{}^k=P_ku_o-Q_kv_o \quad y_o{}^k=Q_ku_o+P_kv_o \cdots\cdots (8).$$

If we substitute these x^ky^k, $x_o{}^ky_o{}^k$ to (5) we get $S=\sqrt{g}e^{i\varphi}(u+iv)$ ·············(9)

In (9) $\sqrt{g}$ means $\sqrt{[PP]+[QQ]}$ where $[PP]=\sum_{K=1}^{2}P_k{}^2$, $[QQ]=\sum_{K=1}^{2}Q_k{}^2$, and $e^{i\varphi}$ is an indeterminate phase factor whose norm is unity. In (9) P, Q are constants, so S is a regular functions of u and v.

§ 2. Trigonometric functions, Triangular area.

On the intersecting angle between two given complex straight line, we have explained already, but we shall explain here systematically again.

Let us put the coordinates of P, P′ to (z^1, z^2), (z'^1, z'^2), and the angle between two complex lines OP and OP′ to θ.

The we can define that

$$\cos\theta=\sum_{K=1}^{2}\frac{z^k}{S}\ \frac{z'^k}{S'} \cdots\cdots (10)$$

where, $\overline{OP}=S$ $\overline{OP}\doteq S'$. If

we put the values of z^k, z'^k in (6) to (10), so the parameters u,u', v,v' are eliminated, then we get

$$\cos\theta=\frac{\{[PP']+[QQ']\}+i\{[PQ']-[P'Q]\}}{\sqrt{g}\ \sqrt{g'}}\ e^{i} \cdots\cdots (11).$$

In the equation (11), $[PP']$, $[QQ']$ are $\sum_{K=1}^{2}P_kP_k'$, $\sum_{K=1}^{2}Q_kQ_k'$ respectively. From (11)

we get $$\|\cos\theta\|=\sqrt{\frac{\{[PP']+[QQ']\}^2+\{[PQ']-[Q'Q]\}^2}{gg'}} \cdots\cdots (12).$$

The equation (12) means that $\|\cos\theta\|$ is coincident perfectly to (7) in the preceding paper.[5)]

From the facts of the above, we get an expression of $\cos\theta$, which is a regular function of parameters, and whose norm has geometric meanings in Unitary space. So the regularization of $\cos\theta$ was accomplished completely.

Then we consider the regularization of triangular areas. Let us consider the triangle OPP′, then we can define its area to the form

$$F=\tfrac{1}{2}\{z^1z'^2-z'^1z^2\} \cdots\cdots (13),$$

from an extention of real space. The equation (13) means that F is a regular function of z^k, z'^k, so if the norm of F has geometrical meanings, we can say that the regularization of area was accomplished.

To ascertain that, we may see the relation

$$2\|F\|=\|S\|S'\|\|\sin\theta. \qquad \cdots\cdots(14).$$

To obtain the relation (14), at first, we get from (1) the relation

$\|S\|=\sqrt{x^{1^2}+y^{1^2}+x^{2^2}+y^{2^2}}\ \|S'\|=\sqrt{x'^{1^2}+y'^{1^2}+x'^{2^2}+y'^{2^2}}$, then by substitution of

$\|\cos\theta\|^2=\dfrac{\{[xx']+[yy]\}^2+\{[xy']-[x'y]\}^2}{S^2S'^2}$, which is equivalent to (12), to the relation $\|\sin\theta\|=\sqrt{1-\|\cos\theta\|^2}$, we get

$$\|\sin\theta\|^2=\frac{S^2S'^2-\{[xx']+[yy']\}^2+\{[xy']-[x'y]\}^2}{S^2S'^2}.$$

Then if we compare the value of ‖F‖, which is reduced from (14), to that of reduced from (13). we can ascertain that these two values of ‖F‖ are coincident completely. So if we define the triangular area OPP′ by (13), we may accomplish the regularization of triangular area. It is to be noticed that the angle which was used here, is not meant the intersecting angle of the vector $\overline{OP}$ and $\overline{OP'}$, but the intersecting angle of the holomorphic plane OP and OP′. Then if three points O, P, P′, are on a same holomorphic plane, the values of $\sin\theta$ is zero, so it is seen that the triangular area is also zero.

Let us rewriten the form of F by the use of parameters. From (13) we get

$$2F=\left\{\begin{vmatrix} x^1x^2 \\ x^1x^2\end{vmatrix}-\begin{vmatrix} y^1\ y^2 \\ y'^1y'^2\end{vmatrix}\right\}+i\left\{\begin{vmatrix} x^1\ y^2 \\ x'^1\ y'^2\end{vmatrix}+\begin{vmatrix} y^1\ x^1 \\ y'^1\ x'^2\end{vmatrix}\right\}. \qquad \cdots\cdots(15).$$

If we put the values of $x^k y^k$ of (6) to (15), we obtain

$$2F=\begin{vmatrix}(P_1+iQ_1)(u+iv) & (P_2+iQ_2)(u+iv) \\ (P_1'+iQ_1')(u'+iv') & (P_2'+iQ_2')(u'+iv')\end{vmatrix}$$

$$=(u+iv)(u'+iv')\left\{\left(\begin{vmatrix}P_1\ P_2 \\ P_1'P_2'\end{vmatrix}-\begin{vmatrix}Q_1\ Q_2 \\ Q_1'Q_2'\end{vmatrix}\right)+i\left(\begin{vmatrix}P_1\ Q_2 \\ P_1'Q_2'\end{vmatrix}+\begin{vmatrix}Q_1\ P_2 \\ Q_1'Q_2'\end{vmatrix}\right)\right\}. \quad \cdots\cdots(16).$$

In (16) we see that F is a regular function of u, v, and u′, v′. At the end of the section we get the regularization of $\sin\theta$ from the relation $\sin\theta=\dfrac{2F}{SS'}$; the meanings of the norm was explained already. So if we substitute the values of *F*, *S*, *S′* from (9) and (16) to the above we get

$$\sin\theta=\frac{\left(\begin{vmatrix}P_1\ P_2 \\ P_1'P_2'\end{vmatrix}-\begin{vmatrix}Q_1\ Q_2 \\ Q_1'P_2'\end{vmatrix}\right)+i\left(\begin{vmatrix}P_1\ P_2 \\ P_1'P_2'\end{vmatrix}+\begin{vmatrix}Q_1\ P_2 \\ Q_1'P_2'\end{vmatrix}\right)}{\sqrt{g}\ \sqrt{g'}} \qquad \cdots\cdots(17).$$

From (17) we may say that the regularization of $\sin\theta$ was accomplished. From these relations we can say that the regularization of some fundamental magnitudes in complex space were accomplished. So we shall consider the regularization of arc length of complex curves in next step.

§ 3. Arc length along a complex curve.

The arc length along a complex cnrve is considered as a limiting case of a straight line. From (9) we get $S=\sqrt{g}e^{i\varphi}(u+iv)$ along a complex straight line. In this case $\sqrt{g}$ is constant, so S is a regular function of u and v, but in general case $\sqrt{g}$ is a function of u and v. So if we find a regular function $U(uv)+iV(uv)$ whose norm is $\sqrt{g}(u^2+v^2)$, so our regularization is accomplished; but it is impossible in general case. So the regularization of arc length along a complex curve must be considered as a limiting case of a complex straight line. We shall reduce the arc length along a complex curve from the metric properties of a holomorphic surface.

Let P, Q be infinitesimally consecutive points on a holomorphic surface. We may put the arc length $\widehat{PQ}$ to ds, and we get the length of a finite arc, by the integration of ds on the holomorphic surface.

To do that, we consider two tangential planes at P and Q respectively; for these tangential planes are holomorphic, their intersecting point is decided uniquely. So, we put the intersecting point to I, and define the infinitesimal arc length $\widehat{PQ}$, as the sum of the infinitesimal distances $\overline{PI}$ and $\overline{IQ}$ on these tangential holomorphic planes. We do not know whether the sum of these infinitesimal distance are regulsr or not. To these concerning facts we get the following theorem.

Theorem 2. *The infinitesimal arc length along a complex curve is given by a regular function of parameters. So the arc length along a infinitesimally closed curve on a holomorphic surface considered to be zero in locally sense.*

To prove the theorem we put the parameters u, v to zero at the given point O on the holomorphic surface. So the equation of the holomorphic surface in the neighbourhood of O are given by

$$z^k=(P_k+iQ_k)(u+iv)+\frac{1}{2}(L_k-iM_k)(u+iv)^2 \qquad \cdots\cdots\cdots(18).$$

In (18) P_k, Q_k, Γ_k, M_k are that

$$\frac{\partial x^k}{\partial u}=\frac{\partial y^k}{\partial v}=P_k \quad \frac{\partial y^k}{\partial u}=-\frac{\partial x^k}{\partial v}=Q_k \qquad \cdots\cdots\cdots(19)$$

$$\frac{\partial^2 x^k}{\partial u_2}=\frac{\partial^2 y^k}{\partial u\partial v}=-\frac{\partial^2 x^k}{\partial v^2}=L_k \quad -\frac{\partial^2 y^k}{\partial u^2}=\frac{\partial^2 x^k}{\partial u\partial v}=\frac{\partial^2 y^k}{\partial v^2}=M_k \quad (k=1.2)$$

in (19) $x^k y^k$ are the real and imaginary part of z^k.

Then the equation (18) are reduced to the form (6)

$$x^k=P_k u-Q_k v+\frac{1}{2}L_k(u^2-v^2)+M_k uv \qquad \cdots\cdots\cdots(20).$$

$$y^k=Q_k u+P_k v-\frac{1}{2}M_k(u^2-v^2)+L_k uv$$

To prove the Theorem 2 on the holomorphic surface (20), we shall prove the the next lemma.

Lemma. *If we put δs to the infinitesimal arc length $\widehat{OA}$ on the holomorphic surface,* *and ds to the infinitesimal distance $\overline{OA}$ on the holomorphic plane which pass through the two given points O and A, so there are the relation*

$$\delta s = \left(1+\frac{1}{2}\left\{{1 \atop 11}\right\}\delta u+\frac{1}{2}\left\{{1 \atop 12}\right\}\delta v\right)ds \quad, \qquad \cdots\cdots(21),$$

where $\left\{{1 \atop 11}\right\}$ and $\left\{{1 \atop 12}\right\}$ are the Christoffel's symbols on the holomorphic surface, and δu, and δv are the differences of the parameters.

To prove the lemma, we shall calculate the arc length $\overset{\frown}{OA}$ by the definition of the above. That is to say, the arc length between O(o, o) and A(δu, δv) on the holomorphic surface is given by $\overline{OI}+\overline{IA}$, where I is the intersecting point of the two tangential planes at O and A. So we shall obtain, at first, the equations of these tangential planes.
In general, the equations of the tangential plane of the surface $x^k=x^k(uv)$ $y^k=y^k(uv)$ $(k=1.2)$ at the point $(u_0\ v_0)$ are given by

$$x^k=x^k(u_0\,v_0)+\left(\frac{\partial x^k}{\partial u}\right)_{\substack{u=u_0\\v=v_0}} u+\left(\frac{\partial x^k}{\partial v}\right)_{\substack{u=u_0\\v=v_0}} v,\quad y^k=y^k(u_0v_0)+\left(\frac{\partial y^k}{\partial u}\right)_{\substack{u=u_0\\v=v_0}} u+\left(\frac{\partial y^k}{\partial v}\right)_{\substack{u=u_0\\v=v_0}} v \quad \cdots(22),$$

where u, v are the parameters on the tangential plane. So from (19) we see that the equations of tangential planes at O is

(I) $x^k=P_ku-Q_kv \quad y^k=Q_ku+P_kv$ $\cdots\cdots(23)$

and the equations of tangential plane at A is

$$\text{(II)}\quad x^k=P_k\delta u-Q_k\delta v+\frac{1}{2}L_k(\delta u^2-\delta v^2)+M_k\delta u\delta v+(P_k+L_k\delta u+M_k\delta v)\bar{u}$$
$$-(Q_k-M_k\delta u+L_k\delta v)\bar{v} \qquad \cdots\cdots(24).$$
$$y^k=Q_k\delta u+P_k\delta v-\frac{1}{2}M_k(\delta u^2-\delta v^2)+L_k\delta u\delta v+(Q_k-M_k\delta u+L_k\delta v)\bar{u}$$
$$+(P_k+L_k\delta u+M_k\delta v)\bar{v}$$

in (24) $\bar{u}$, $\bar{v}$ are the parameters on the plane (11).

To obtain the coordinates of the intersecting point I, we shall get the values of parameters u, v and $\bar{u}$, $\bar{v}$, which give the same values of x^k, y^k from the equations of (I) and (II), These values of parameters evidently are

$$u=\frac{\delta u}{2}\quad v=\frac{\delta v}{2},\quad \bar{u}=-\frac{\delta u}{2},\quad \bar{v}=-\frac{\delta v}{2}. \qquad \cdots\cdots(25).$$

From the equations of (I), (II) and (25) we get

$$\overline{OI}=\sqrt{g}\left(\frac{\delta u}{2}+i\frac{\delta v}{2}\right). \qquad \cdots\cdots(26).$$

To obtain $\overline{IA}$ after the rejection of higher terms of δu, δv in

$$\sqrt{\sum(P_k+L_k\delta u+M_k\delta v)^2+\sum(Q_k-M_k\delta u+L_k\delta v)^2},$$

we get

$$\overline{IA}=\sqrt{g}\left\{1+\frac{[PL]-[QM]}{g}\delta u+\frac{[PM]+[QL]}{g}\delta v\right\},$$

then we get

$$\widehat{OA}=\overline{OI}+\overline{IA}$$

$$=\sqrt{g}(\delta u+i\,\delta v)+\sqrt{g}\left\{\frac{[PL]-[QM]}{g}\delta u+\frac{[PM]+[QL]}{g}\delta v\right\}\left(\frac{\delta u}{2}+i\frac{\delta v}{2}\right). \quad\cdots\cdots\cdots\cdots(27).$$

On the other hand we see that the fundamental tensors of the holomorphic surfaces (20) is

$$g^{11}=g^{22}=\frac{1}{g}\quad g_{11}=g_{22}=g \quad\cdots\cdots\cdots\cdots(28)$$

then we get

$$\frac{\partial g_{11}}{\partial u}=\frac{\partial g_{22}}{\partial u}=2\{[PL]-[QM]\},\quad \frac{\partial g_{11}}{\partial v}=\frac{\partial g_{22}}{\partial v}=2\{[PM]+[QL]\} \quad\cdots\cdots\cdots(29).$$

From (29) we get

$$\begin{Bmatrix}1\\11\end{Bmatrix}=\frac{[PL]-[QM]}{g}\qquad \begin{Bmatrix}1\\12\end{Bmatrix}=\frac{[PM]+[QL]}{g} \quad\cdots\cdots\cdots\cdots(30).$$

If we put $\widehat{OA}=\overline{OI}+\overline{IA}=\delta s$, $\sqrt{g}\,(\delta u+i\delta v)=ds$, and substitute the values of (30) to the equation (27), we see that the equation (21) is $\delta s=\left(1+\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}\delta u+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}\delta v\right)$.

So the lemma is proved.

From the above lemma, we can prove the Theorem 2 easily. Let us take on the holomorphic surface (20), the three infinitesimally consecutive points O(o, o) $A(\delta u, \delta v)$ B $(\delta u', \delta v')$, then we show that the arc length of closed curve OABO is a second ordered infinitesimal with respect to δu and δv.

If we put the arc length $\widehat{OA}$ $\widehat{AB}$ $\widehat{BO}$ to δs_1, δs_2, δs_3 respectively, we get similarly to (21), the followings

$$\delta s_1=\left\{1+\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}\delta u+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}\delta v\right\}ds$$

$$\delta s_2=-\left\{1+\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}(\delta u+\delta' u)+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}(\delta v+\delta' v)\right\}ds$$
$$+\left\{1+\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}(\delta u+\delta' u)\frac{1}{2}\begin{Bmatrix}I\\12\end{Bmatrix}(\delta v+\delta' v)\right\}ds' \quad\cdots\cdots\cdots\cdots(31),$$

$$\delta s_3=-\left\{1+\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}\delta' u+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}\delta' v\right\}ds'$$

in the equation (31) ds and ds' are given by

$$ds=\sqrt{g}\,(\delta u+i\,\delta v),\quad ds'=\sqrt{g}\,(\delta' u+i\,\delta' v)\,. \quad\cdots\cdots\cdots\cdots(32)$$

From (31) evidently we have

$$\delta s_1+\delta s_2+\delta s_3=\left\{\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}\delta u+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}\delta v\right\}ds'$$
$$-\left\{\frac{1}{2}\begin{Bmatrix}1\\11\end{Bmatrix}\delta' u+\frac{1}{2}\begin{Bmatrix}1\\12\end{Bmatrix}\delta' v\right\}ds\,.$$

If we put the values of ds′ in (32) to the above we get

$$\delta s_1+\delta s_2+\delta s_3=\frac{\sqrt{g}}{2}\left(\left\{\begin{matrix}1\\12\end{matrix}\right\}-i\left\{\begin{matrix}1\\11\end{matrix}\right\}\right)(\delta' u\delta v-\delta u\delta' v) \qquad \cdots\cdots(33),$$

From the results of (33) Theorem 2 is proved.

The results of the above seems very likely to the relation of Euclidean space and its connected space. That is in Euclidean space, the direction of a vector returns back to that of the initial position, after the parallel displacement along a finife closed curve; but in the connected space it returns back in the sense of local properties, and it does not, after a finite displacement. So we can say that the theories of regular functions on Gauss's plane correspond to the theories of gemometry in Euclidean space, and the theories of regular functions on the holomorphic surfaces correspond to the theories of geometry in its connected space.

§ 4. Curvature of a complex curve.

We can define the curvature of a complex curve at O, thus.

$$\frac{1}{\rho}=\lim_{A\to 0}\frac{\delta\theta}{\delta s}. \qquad \cdots\cdots(34),$$

In (34) O(o, o) and A $(\delta u, \delta s)$ are two given consecutive points, and δs is the arc length of $\widehat{OA}$, and $\delta\theta$ is the intersecting angle of the tangential planes at O and A respectively. Because of $\delta\theta$ is infinitesimal we can substitute the value of $\sin\delta\theta$ to $\delta\theta$ in (34). From (23) (24) (17) we get

$$\sin\delta\theta=\frac{1}{g}\{(a\delta u-b\delta v)+i\,(b\delta u+a\delta v)\} \qquad \cdots\cdots(35),$$

where a, b are that

$a=P_1L_2-P_2L_1+Q_1M_2-Q_2M_1 \quad b=-P_1M_2+P_2M_1-Q_2L_1+Q_1L_2$ respectively.

Because of the facts that δs is infinitesimal, we can put the value of $\delta s=\sqrt{g}(\delta u+i\delta v)$ to δs in (34), so we get

$$\frac{1}{\rho}=\frac{(a\delta u-b\delta v)+i\,(b\delta u+a\delta v)}{g^{3/2}\,(\delta u+i\,\delta v)}=\frac{(a+ib)}{g^{3/2}} \qquad \cdots\cdots(36).$$

So if the geometrical meanings of the norm of $\frac{1}{\rho}$ are given, from (36) the regularization of $\frac{1}{\rho}$ was accomplished. Concerning the facts, we see that the next theorem is hold.

Theorem 3. *The norm of curvature of a complex curve at any giaon point, is equal to $\sqrt{-\frac{1}{2}K}$, where K is the Gauss's total curvature of the holomorphic surface at the given corresponding point.*

Proof. If we construct the Riemann Christoffel's tensor on the holomorphic surface (20), from (28) and (29) we get

$$R^{1}_{212}=\frac{2\{[LL]+[MM]\}\{[PP]+[QQ]\}-2\{[PL]-[QM]\}^2-2\{[PM]+QL]\}^2}{g^2} \quad \cdots(37).$$

Simplifying the numerator of the above we can reduce it to the form $2(a^2+b^2)$ On the other hand the Gauss's total curvature K is redused to

$$K=-\frac{R_{1212}}{g_{11}g_{22}}=-\frac{g_{11}R^{1}_{212}}{g_{11}g_{22}}=-\frac{R^{1}_{212}}{g_{22}}=\frac{-2(a^2+b^2)}{g^3} \quad \cdots\cdots\cdots\cdots\cdots(38).$$

Comparing (38) to (36) we get $\left\|\frac{1}{\rho}\right\|=\sqrt{-\frac{1}{2}K}$ $\cdots\cdots\cdots\cdots\cdots\cdots(39)$,

then the theorem was proved.

It is to be noticed that the Gauss's total curvatue of holomorphic surface is a negative quantiiy-

Conclusions.

In this paper we have discussed the metric properties of two dimentional complex space only, then we expect that we refer to the metric properties of higher dimentional complex space in the next paper.

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