Remarks on the Convexity of Connected Sets.

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The set M in the separable real Banach space Ω [1] is said to be locally convex if every point x of the closure $\bar{M}$ is a convex point, that is, there is a positive number δ such that $U_\varepsilon(x) \cap M$[1], if non-null, is convex for any positive $\varepsilon \leqq \delta$. We have proved already in [2] that if M is locally convex and arcweise connected, its interior M^i is cnvex.

In this note we shall introduce the concept of homogeneity into local convex sets to lay more somewhat firmly the foundation of our theory and then shall give the proof of main theorem by discussing the case where M is closed under the weaker condition and some assumption.

1. Preliminaries.

We denote by $S(M)$ the subspace[2] spanned by M i.e., the minimal subspace of Ω which contains M. If $S(U_\varepsilon(x) \cap M) = S(U_{\varepsilon'}(y) \cap M)$ for every different $x, y \in \bar{M}$ and every positive numbers ε, ε', then M is said to be homogeneous. This definition is equivalent to the equality that $S(U_\varepsilon(x) \cap M) = S(M)$ for every $x \in M$ and every $\varepsilon > 0$.

Lemma 1. *If M is convex, then M is homogeneous.*

Proof. Let $U_\varepsilon(a)$ be a neighbourhood of a point $a \in \bar{M}$.

Suppose $S(U_\varepsilon(a) \cap M) \subsetneqq S(M)$, then there is a point $b \in S(M) - S(U_\varepsilon(a) \cap M)$. It can be easily seen that $f(\lambda) = (1-\lambda)a + \lambda b$, $0 < \lambda < \frac{\varepsilon}{\|a-b\|}$, belongs to $S(M) - S(U_\varepsilon(a) \cap M)$. This contradicts to the definition of $S(U_\varepsilon(a) \cap M)$.

More precisely,

Lemma 2. *If M is connected and locally convex, then M is homogeneous.*

Proof. Suppose that the theorem is not true. Let $U_\varepsilon(a)$ be a neighbourhood of a point $a \in \bar{M}$. We denote by X the set of points x belonging to $\bar{M}$ which satisfy $S(U_\delta(x) \cap M) = S(U_\varepsilon(a) \cap M)$ for every δ and fixed ε. Let $Y = \bar{M} - X$. Then $X \neq 0$, and $Y \neq 0$. Since M is connected, $\bar{M}$ is also so, that is, $X' \cap Y \neq 0$[3] or $X \cap Y' \neq 0$, where we take up $X' \cap Y \neq 0$ (similarly in the other case). Let $b \in X' \cap Y$. We can choose a neighbourhood $V(b)$ which contains $x \in X$ and whose intersection with M is convex. By Lemma 1, $V(b) \cap M$ is homogeneous, that is, $S(U_\delta(x) \cap M) = S(U_\zeta(b) \cap M)$; accordingly we have $S(U_\eta(a) \cap M) = S(U_\zeta(b) \cap M)$ for any η, $\zeta > 0$, contradicting to the assumption that $b \in Y$. Lemma 2 has thus been proved.

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1) $U_\varepsilon(x) = E[z;\ \|z-x\| < \varepsilon]$.
2) Here the subspace means the linear subspace.
3) X' is the derived set of X.

By a relative interior (exterior) point of M, hereafter, we mean the interior (exterior) point[4] of M in the subspace $S(M)$. The set of all relative interior (exterior) points of M is denoted by M^i (M^e) which is called the relative interior (exterior) of M.

Now we suppose that convex sets always contain relative interior points [3]. Then we have readily

Lemma 3. *If M is connected and locally convex, then it holds that $\overline{M}=\overline{M^i}$.*

Proof. Clearly $\overline{M}\supset\overline{M^i}$. We now show $\overline{M}\subset\overline{M^i}$. Taking any $x\in\overline{M}$, $U_\varepsilon(x)\cap M$ is convex for some $\varepsilon>0$. By the presupposition and Lemma 2, there are $x_0\in U_\varepsilon(x)\cap M$ and $\delta>0$ such that

$$S(M)\cap U_\delta(x_0)=S(U_\varepsilon(x)\cap M)\cap U_\delta(x_0)\subset U_\varepsilon(x)\cap M\subset M.$$

Hence we have $\overline{M}\subset\overline{M^i}$.

Further we arrange three lemmas [4].

Lemma 4. *Let a, $b\in\Omega$* and $c=(1-\lambda)a+\lambda b$. *Given a neighbourhood $U(c)$, we can find $V(b)$ such that $[a, x]\cap U(c)\neq 0$*[5] *for any $x\in V(b)$.*

Lemma 5. *Let M be a convex set, and M^* its bounday. If $a\in M^i$, $r\in M^*$, then $\overrightarrow{(a, r)}\subset M^e$ where $\overrightarrow{(a, r)}=[z\,;\ z=(1-\lambda)a+\lambda r,\ \lambda>1]$*

Lemmn 6. *Let b be a convex point of M and let $[a, b)\subset M^i$. Then there is a suitable neghbourhood $U(b)$ of b such that $[a, x)\subset M^i$ for every $x\in U(b)\cap M$.*

Lemma 7. *If M is connected and locally convex, then $(\overline{M})^i=M^i$.*

Proof. We can show easily that any boundary point of M has relative interior points and relative exterior points of M in its arbitrary neighbourhoods. Therefore we have $(\overline{M})^i\subset M^i$. Clearly $(\overline{M})^i\supset M^i$.

Lemma 8. *The relative interior of a convex set is also convex* [5].

2. Main Theorem.

We shall have

Theorem 1. *If M is connected and locally convex, then its relative interior M^i is convex.*

In order to prove Theorem 1, it is sufficient that we prove Theorem 1′ as follows.

Theorem 1′ *If M is closed connected and locally convex, then M is convex.*

In fact, it is readily seen that the two hold: (1) if M is connected, $\overline{M}$ is connected, (2) if M is locally convex, $\overline{M}$ is locally convex. If $\overline{M}$ is proved to be convex by Theorem 1′, it will be concluded that M^i is convex by Lemma 7 and 8.

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4) Precisely, the point x is called a relative interior point if $S(M)\cap U_\varepsilon(x)\subset M$ for some $\varepsilon>0$, and x a relative exterior point if $S(M)\cap U_\varepsilon(x)\subset CM$ (complement of M).

5) The notations $[a, x]$, $[a, x)$ are defined in [1] p. 26.

Proof of Theorem 1'.

Suppose that M is not convex, then there are points a, $b \in M$ such that $[a, b] \not\subset M$, that is, $[a, b]$ contains a relative exterior point of M, for M is closed. Without restriction the point b may be considered to be a relative interior point of M by Lemma 3 and 4.

Let X be the set composed of all points x of M such that $[b, x] \subset M$ and let Y be $M-X$. Then $M=X \cup Y$, $X \neq O$, $Y \neq O$. Since M is connected, it holds that either $X' \cap Y \neq O$ or $X \cap Y' \neq O$; let, for instance, $r \in X \cap Y'$ (being similar in other case). Then $[b, r] \subset M$ and a neighbourhood $U_\xi(r)$ contains a point $p \in Y$ yielding $[b, p] \not\subset M$, while $U_\xi(r)$ can be chosen so that $U_\xi(r) \cap M$ is coevex. Now every point of $[r, p]$ included by M is represented as $f(\lambda)=(1-\lambda)p+\lambda r$, $0 \leqq \lambda \leqq 1$.

Letting $\lambda_0 = sup[\lambda \,|\, [b, f(\mu)] \not\subset M \text{ for every } \mu,\ 0<\mu<\lambda \leqq 1$, it follows that a boundary point of M lies in $(b, f(\lambda_0))$. Because if it held that $[b, f(\lambda_0)) \subset M^i$, there would exist some neighbourhood $U_\delta(f(\lambda_0))$ such that $[b, x) \subset M^i$ for any $x \in U_\delta(f(\lambda_0)) \cap M$ by Lemma 6. Consequently it would follow that $[b, f(\mu)] \subset M$ whenever $\lambda_0 - \varepsilon < \mu < \lambda_0$ for an adequate $\varepsilon > 0$. This contradicts with the assumption of λ_0.

Let $g(\nu)=(1-\nu)b+\nu f(\lambda_0)$ be every point of $[b, f(\lambda_0)]$ and $\nu_0 = inf[\nu \,|\, g(\nu) \in M^*]$. Then $g(\nu_0)$ is clearly a boundary point and $V \cap M$ is convex for an adequate neighbourhood V of $g(\nu_0)$. There is $\eta > 0$ such that $g(\nu) \in V$ for every ν, $\nu_0 - \eta < \nu < \nu_0 + \eta$; in detail, the points $g(\nu')$ are interior points of $M \cap V$ for ν', $\nu_0 - \eta < \nu' < \upsilon_0$ and $g(\nu'')$ are exterior points of $M \cap V$ for ν'', $\nu_0 < \nu'' < \nu_0 + \eta$ by Lemma 5, accordingly $g(\nu'')$ has its neighbourhood $W \subset M'$. Utilizing Lemma 4, it holds that $[b, y] \cap W \neq O$ for every $y \in N(f(\lambda_0); \delta)$ where $N(f(\lambda_0); \delta)$ is some neighbourhood of $f(\lambda_0)$. Especially for some ζ_0, we have $[b, f(\lambda_0+\delta)] \cap W \neq O$, $0<\zeta<\zeta'$, contradicting with the definition of λ_0. Thus the proof of the theorem has been completed.

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Notes.

〔1〕 Separability of Ω exerts an effect on Lemma 6. With respect to this, see

T. TAMURA, *On a relation between local convexity and entire convexity*, Journal of Science of the Gakugei Faculty, Tokushima University, Vol. 1, (1950) p. 30 and p. 27.

〔2〕 The paper 〔1〕, pp. 25—30.

〔3〕 This supposition is serious, but we don't here treat it in detail. Of course, it holds naturally when Ω is finite dimensional.

〔4〕 Lemma 4, 5, and 6 equal to Corollary 1 (p. 26), Corollary 2 (p. 27), and Lemma 3 (p. 27) respectively of this Journal Vol. 1 (1950).

〔5〕 This is proved easily by the preceeding lemmas.

Addendum.

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