Note on Inverses in Rings.

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Let R be an arbitrary ring. Two idempotent elements e and f are called isomorphic in R if there exist two elements a and b such that $ab=e$ and $ba=f$. We write then $e\cong f$. Clearly, by this definition, the set of all idempotent elements in R are divided into classes of idempotent elements. By [1][1], we may assume in the above definition that $a \in eRf$ and $b \in fRe$.

Lemma 1. *If an idempotent element e is isomorphic to zero, then $e=0$.*

Lemma 2. *If two idempotent elements e and f are isomorphic and if e is a sum of two orthogonal idempotent elements e_1 and e_2, then $f=f_1+f_2$, where f_1 and f_2 are orthogonal idempotent elements such that $f_1\cong e_1$ and $f_2\cong e_2$.*

Proof. From $e=ab$ and $f=ba$, we have $be=fb$ and so $bea=fba=f$. Hence $f=bea =be_1a+be_2a$. We see easily that be_1a and be_2a are orthogonal idempotent elements such that $be_1a\cong e_1$ and $be_2a\cong e_2$.

Lemma 3. (Azumaya [1]). *Two idempotent elements e and f are isomorphic if and only if the left ideals Re and Rf are R-isomorphic: $Re\cong Rf$.*

Lemma 4. *If the left ideals Re and Rf are R-isomorphic, then the subrings eRe and fRf are isomorphic.*

Proof. From $e=ab$ and $f=ba$ ($a \in eRf$, $b \in fRe$), we see that eRe and fRf are isomorphic under the mapping $x \longrightarrow bxa$ ($x \in eRe$).

In the following we assume that R contains an identity 1. Generally, $ab=1$ in R does not imply $ba=1$.

Lemma 5. *The following conditions are equivalent:*

(i) *$ab=1$ in R implies always $ba=1$.*

(ii) *If $e\cong 1$, then $e=1$.*

(iii) *For any idempotent element $e\neq 1$, R and Re are not R-isomorphic.*

Proof. Suppose that R contains a pair of elements a, b such that $ab=1$ but $ba\neq 1$. Then ba is an idempotent element and $ba\cong 1$.

Theorem 1. *If R is a ring with an identity that contains two elements a and b such that $ab=1$, $ba\neq 1$, then*

(i) *R contains an infinite number of idempotent elements e_i such that $e_i\cong 1$.*

(ii) *R contains an infinite number of subrings R_i such that the R_i are isomorphic to R.*

(iii) *R contains a left ideal that is a direct sum of an infinite number of R-isomorphic left ideals* [2].

(iv) *There exists an infinite properly descending chain of principal left ideals generated by idempotent elements:*

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1) Numbers in brackets refer to the references at the end of the paper.

$$R = Re_0 \supset Re_1 \supset Re_2 \supset \cdots\cdots \qquad (e_0 = 1)$$

such that the factor spaces Re_i/Re_{i+1} $(i=0,1,2,\cdots\cdots)$ *are R-isomorphic.*

Proof. Since ba is an idempotent element, we have $1=(1-ba)+ba$, where $1-ba$ and ba are orthogonal idempotent elements. Then we have by Lemma 2 $ba=(ba-b^2a^2)+b^2a^2$, where $ba-b^2a^2$ and b^2a^2 are orthogonal idempotent elements and so

$$1=(1-ba)+(ba-b^2a^2)+b^2a^2 ,$$

where

$$1-ba \cong ba-b^2a^2 , \qquad 1 \cong ba \cong b^2a^2 .$$

Continuing in this way, we have for any positive integer n

$$1=\sum_{i=1}^{n}(b^{i-1}a^{i-1}-b^ia^i)+b^na^n \qquad (a^0=b^0=1),$$

where

$$1 \cong ba \cong b^2a^2 \cong \cdots\cdots \cong b^na^n ,$$

$$1-ba \cong ba-b^2a^2 \cong \cdots\cdots \cong b^{n-1}a^{n-1}-b^na^n .$$

Since $1-ba \neq 0$, we see from Lemma 1 that $b^{i-1}a^{i-1}-b^ia^i \neq 0$ and so $b^ia^i \neq b^ja^j$ $(i \neq j)$. Hence there exists an infinite number of idempotent elements b^ia^i such that $b^ia^i \cong 1$. If we set $b^ia^i=e_i$, then

$$R \supset e_1Re_1 \supset e_2Re_2 \supset \cdots\cdots$$

and by Lemma 4

$$R \cong e_iRe_i \qquad (i=1,2,3,\cdots\cdots).$$

Further if we set $b^{i-1}a^{i-1}-b^ia^i=f_i$, then the left ideal $\mathfrak{l}=\sum_i Rf_i$ is a direct sum of R-isomorphic left ideals Rf_i. Finally

$$R=Re_0 \supset Re_1 \supset Re_2 \supset \cdots\cdots$$

is an infinite properly descending chain of left ideals such that $Re_i/Re_{i+1} \cong Rf_i$.

For example, let R be a (complete) direct sum of an infinite number of simple algebras. Then, by Theorem 1 (iv), we see that $ab=1$ in R always implies $ba=1$.

Theorem 2. *Let R be a ring with an identity. Suppose that R splits into a direct sum of an infinite number of left ideals:*

$$R=\mathfrak{l}_1+\mathfrak{l}_2+\mathfrak{l}_3+\cdots\cdots$$

If R contains an infinite number of R-isomorphic left ideals $\mathfrak{l}_i$, *then there exists a pair of elements a, b in R such that* $ab=1$ *but* $ba \neq 1$.

Proof. We see easily that every $\mathfrak{l}_i$ is a left ideal generated by an idempotent element: $\mathfrak{l}_i=Re_i$. Let us assume

$$Re_{i_1} \cong Re_{i_2} \cong Re_{i_3} \cong \cdots\cdots$$

We may assume without loss of generality that $e_{i_1}=e_1$. Let $Re_{i_{k-1}}$ be R-isomorphic to Re_{i_k} under the mapping $a_{i_{k-1}} \longrightarrow a'_{i_k}$. Then the mapping

$$a=\sum_{i=1}a_i \longrightarrow a^*=\sum_{i=2}a_i^* \qquad (a_i,\ a_i^* \in Re_i),$$

where $a_i^*=a_i$ if $i \neq i_k$ $(k=1,2,\cdots\cdots)$ and $a_{i_k}^*=a'_{i_k}$ $(k=2,3,\cdots\cdots)$, gives the R-isomorphism between R and $R(1-e_1)$. Hence, by Lemma 5, R contains a pair of elements a, b such that $ab=1$ but $ba \neq 1$.

We denote by $r(a)$ [$l(a)$] the right [left] annihilator of an element a.

Lemma 6. *If* $ab=1$, *then* $l(a)=0$ *and* $r(a)=r(ba)=(1-ba)R$.

Lemma 7. *If an element a of a ring with an identity has a unique right inverse b, then b is the inverse of a.*

Proof. Since $a(b+1-ba)=1$, we have $1-ba=0$.

Theorem 3 (Jacobson [2]). *If R contains two elements a and b such that* $ab=1$ *but* $ba\neq 1$, *then the element a has an infinite number of right inverses.*

Proof. Since $l(a)=0$, we have $a^i\neq 0$ $(i=1, 2, \cdots\cdots)$ and so $(1-ba)a^i\neq 0$. Moreover we have $1-ba\neq(1-ba)a^i$ because $((1-ba)a^i)^2=0$. Hence

$$(1-ba)a^i\neq(1-ba)a^j \qquad (i\neq j).$$

If we set $b_i=b+(1-ba)a^i$, then $ab_i=1$ $(i=0, 1, 2, \cdots\cdots)$[2].

If b and b' are two distinct right inverses of a, then, by Lemma 6, $ba\neq b'a$. But we have

$$Rb'a=Ra=Rba \ .$$

References

[1] G. AZUMAYA, On generalized semi-primary rings and Krull-Remak-Schmidt's theorem, Jap. J. Math., Vol. 19 (1948).

[2] N. JACOBSON, Some remarks on one-sided inverses, Proc. Amer. Math. Soc., Vol. 1 (1950).

2) The b_i thus constructed are equal to those in [2], Theorem 3.