Notes on General Analysis (II)

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In these notes, we shall first state of the necessary and sufficient conditions that a function should be homogeneous polynomials of degree n, in § 1. In § 2, we shall investigate whether some of the theorems of Schwarz on regular functions of complex variables will be able to be extended to the case of functions whose domain and range both lie in complex-Banach-spaces or not. Finally, in § 3, we shall investigate the state of the boundary of the domain $G(h_k)$.

§ 1. Homogeneous polynomials.

Let E and E' be two complex-Banach-spaces.

Definition 1.[*)] *An E'-valued function $x'=p(x)$ defined on E is called a homogeneous polynomial of degree n, if the following conditions are satisfied: (1) $p(x)$ is strongly continuous at each point of E, (2) for each x and y in E, and for any complex number a, $p(x+ay)$ can be expressed as $p(x+ay)=\sum_{k=0}^{n} p_k(x, y)a^k$, where $P_k(x,y)$ are arbitrary E'-valued functions of two variables x and y, (3) $P_n(x, y)\neq O$ for some x and y, (4) $p(ax)=a^n p(x)$.*

Definition 2.[*)] *An E'-valued function $x'=f(x)$ defined on a domain D of E is called analytic, if it is strongly continuous and G-differentiable on D.*

Theorem 1. *The necessary and sufficient conditions that $p(x)$ should be a homogeneous polynomial of degree n are that it is analytic on E and satisfies $p(ax)=a^n p(x)$.*

Proof. If p(x) is a homogeneous polynomial of degree n, it satisfies $p(\alpha x)=\alpha^n p(x)$ by (4) and is strongly continuous by (1). The condition (2) shows that p(x) is G-differentiable at any point in E. Thus we see that the conditions are necessary. Conversely, let p(x) be analytic at any point of E and satisfies $p(\alpha x)=\alpha^n p(x)$. Then we have

$$P(x+\alpha y)=\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi-\alpha}\, d\xi=\frac{1}{2\pi i}\int_c\left(\sum_{m=0}^{\infty}\frac{P(x+\xi y)}{\xi^{m+1}}\alpha^m\right)d\xi$$

$$=\sum_{m=0}^{\infty}\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\alpha^m d\xi$$

$$=\sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}d\xi\right)\alpha^m ,$$

where C is a circle of radius r and $r > |\alpha|$.

While, $p(\alpha x)=\alpha^n p(x)$, we have

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\, d\xi = \frac{1}{2\pi i}\int_c \frac{P((1/\xi)x+y)}{\xi^{m+1-n}}\, d\xi$$

Put $\frac{1}{\xi}=\eta$, then $d\xi=-\frac{1}{\eta^2}d\eta$ and $\frac{1}{\xi^{m+1-n}}=\eta^{m-n+1}$

Therefore, we have

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\,d\xi=-\frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\,d\eta\,.$$

Since the left side integral is taken counterclockwise along the circle $|\xi|=r$, the right side integral is taken clockwise along the circle $|\eta|=\frac{1}{r}$. Integrating counterclockwise along the circle $|\eta|=\frac{1}{r}$, we have

$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\,d\xi=\frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\,d\eta\,.$$

Since $p(\eta x+y)$ is regular on $|\eta|\leqq\frac{1}{r}$,

$$\frac{1}{2\pi i}\int_{c'} P(\eta x+y)\eta^{m-n-1}\,d\eta=P(y)\,,\quad \text{when}\quad m=n\,,$$

$$=0\,,\quad \text{for}\quad m>n+1\,.$$

Put
$$\frac{1}{2\pi i}\int_c \frac{P(x+\xi y)}{\xi^{m+1}}\,d\xi=P_m(x,y)\,,$$

then we have
$$P(x+\alpha y)=\sum_{m=0}^{n} P_m(x,y)\alpha^m\,.$$

Since $p_n(x, y)=p(y)$, $p_n(x, y)\not\equiv 0$. This completes the proof.

Corollary. *The necessary and sufficient condition that $\mu(x)$ should be linear is that $\mu(x)$ is analytic on E and satisfies $\mu(ax)=a\mu(x)$.*

§ 2. Extension of the Schwarz's theorem.

The purpose of this chapter is to extend the Schwarz's theorem of complex variables to the case of complex-Banach-spaces. The theorem of Schwarz is described as follows: If f(z) is regular in the circle $|z|<R$ and satisfies $f(0)=0$ and $|f(z)|\leqq M$ in the circle $|z|<R$, then $|f(z)|\leqq\frac{M}{R}|z|$. If the equality is established at a point of $|z|<R$, then $f(z)\equiv\frac{M}{R}e^{i\theta}z$.

This theorem is not always true in our cases.

Theorem 2. *Let an E'-valued function $f(x)$ defined in the sphere $\|x\|<R$ be analytic and satisfies $f(0)=0$ and $\|f(x)\|\leqq M$ in the sphere $\|x\|<R$. Then $\|f(x)\|\leqq\frac{M}{R}\|x\|$*

Proof. Since f(x) is analytic in the sphere $\|x\|<R$ and $f(0)=0$, we have

$$f(x)=\sum_{n=1}^{\infty} h_n(x)\,, \quad \cdots\cdots\cdots(1)$$

for an arbitrary x in $\|x\|<R$, where $h_n(x)$ is a homogeneous polynomial of degree

n. Now, we fix x in $\|x\|<R$. From (1), we have

$$f(\alpha x) = \sum_{n=1}^{\infty} h_n(x)\alpha^n .$$

$f(\alpha x)$ is analytic about α, when $|\alpha|<\frac{R}{\|x\|}$, where clearly $\frac{R}{\|x\|}>1$. Since $f(\alpha x)$ is an analytic function of α, $\frac{f(\alpha x)}{\alpha}=\sum_{n=1}^{\infty}h_n(x)\alpha^{n-1}$ is also analytic in the circle $|\alpha|<\frac{R}{\|x\|}$. Let r be an arbitrary positive number which satisfies $r<\frac{R}{\|x\|}$, then $\|\frac{f(\alpha x)}{\alpha}\|\leqq\frac{M}{r}$, when $|\alpha|=r$, because $\|f(\alpha x)\|\leqq M$ and $|\alpha|=r$. Since $\|\frac{f(\alpha x)}{\alpha}\|$ is subharmonic as to α, $\|\frac{f(\alpha x)}{\alpha}\|$ takes its maximum on $\|\alpha\|=r$. Thus we see that $\|\frac{f(\alpha x)}{\alpha}\|\leqq\frac{M}{r}$, for $|\alpha|\leqq r$. Since r is an arbitrary positive number satisfying $r<\frac{R}{\|x\|}$, we have $\|\frac{f(\alpha x)}{\alpha}\| \leqq M\Big/\frac{R}{\|x\|}=\frac{M}{R}\|x\|$,

for $|\alpha|<\frac{R}{\|x\|}$ (>1). Put $\alpha=1$, and we have

$$\|f(x)\|\leqq \frac{M}{R}\|x\| . \qquad \cdots\cdots(2)$$

Since x is an arbitrary point in $\|x\|<R$, (2) is held for $\|x\|<R$. This completes the proof.

In concluding this paragraph, we shall aford an example f(x) which satisfies following conditions

(1) $f(0)=0$,
(2) $f(x)$ is analytic on $\|x\|<1$,
(3) $\|f(x)\|\leqq M$ on $\|x\|<1$,
(4) $\|f(x)\|=M\|x\|$ for some points in $\|x\|<1$,

and yet $\|f(x)\|\neq M\|x\|$

Let $X=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$ be a matrix of (2, 2)-type of complex numbers, and $\|X\|=Max(|x_{11}|, |x_{12}|, |x_{21}|, |x_{22}|)$. Then the set of such X is clearly complex-Banach-spaces. Put $\mu(X)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}X$ where $\infty>a>b>c>d>0$, and $M=a+b$. Then

$$\mu(X)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}=\begin{pmatrix} a x_{11}+b x_{21} & a x_{12}+b x_{22} \\ c x_{11}+d x_{21} & c x_{12}+d x_{22}\end{pmatrix} .$$

Clearly $\mu(X)$ is a linear function and we see that $\mu(o)=0$ and $\mu(X)$ is an analytic function on whole spaces by Corollary of Theorem 1. Since $\sup_{\|X\|=1}\|\mu(X)\|=a+b=M$, we have $\|\mu(X)\|\leqq M$, when $\|X\|\leqq 1$. Put $X_1=\begin{pmatrix} \lambda & o \\ \lambda & \lambda\end{pmatrix}$ and $X_2=\begin{pmatrix} \lambda & o \\ o & \lambda\end{pmatrix}$, where $0<\lambda<1$.

Then $\|X_1\|=\lambda$ and $\|X_2\|=\lambda$. Since

$$\mu(X_1)==\begin{pmatrix} a\lambda+b\lambda & b\lambda \\ c\lambda+d\lambda & d\lambda \end{pmatrix}, \quad \|\mu(X_1)\|=\lambda(a+b)=M\cdot\|X_1\|.$$

While $\mu(X_2)=\begin{pmatrix} a\lambda & b\lambda \\ c\lambda & d\lambda \end{pmatrix}$ and we see that

$$\|\mu(X_2)\|=\lambda a<\lambda(a+b)=\|X_2\|\cdot M.$$

§ 3. On the boundary of G(hn)**).

Definition 3. *$G(h_n)$ is the interior of the region of convergence of a power series* $\sum_{n=0}^{\infty} h_n(x)$.

Definition 4. *Let x be an arbitrary point on $\|x\|=1$. $R(x)$ is the upper-bound of $|\alpha|$,* for which *$\sum_{n=0}^{\infty} h_n(\alpha x)$ is convergent and analytic at αx.*

Theorem 3. *If $|\alpha|=R(x)$, αx is the boundary point of $G(h_n)$.*

Proof. Since $\sum_{n=0}^{\infty} h_n(x)$ is analytic in $G(h_n)$**), $\sum_{n=0}^{\infty} h_n(x)$ is analytic at αx while αx lies in $G(h_n)$***), where $\|x\|=1$. But $\sum_{n=0}^{\infty} h_n(\alpha x)$ is not analytic when αx lies beyond $G(h_n)$, because $\sum_{n=0}^{\infty} h_n(x)$ does not always converge in any neighbourhood of αx. This proves that αx is a boundary point of $G(h_n)$.

Theorem 4. *$R(x)$ is lower semi-continuous on $\|x\|=1$.*

Proof. If $R(x)$ is not lower semi-continuous at a point x_0 on $\|x\|=1$, there exists a sequence $\{x_i\}$ such that x_i tends to x_0 and satisfies

$$R(x_i) < R(x_0)-\epsilon \qquad (i=1,2,3,\dots),$$

for a suitable positive number ϵ. While, if $|\alpha|=R(x_i)$, there exists at least a point α_i on $|\alpha|=R(x_i)$ such that $\alpha_i x_i$ is a singular point of $\sum_{n=0}^{\infty} h_n(x)$. Since $|\alpha_i|=R(x_i)<R(x_0)-\epsilon$, $\{\alpha_i\}$ has at least a limiting point α_0. Then we have a subsequence $\{\alpha_{i'}\}$ of $\{\alpha_i\}$ which converges to α_0. Thus we see that $\alpha_{i'}x_{i'}$ converges to $\alpha_0 x_0$. Since $\alpha_0 x_0$ is a limiting point of singular points $\alpha_{i'}x_{i'}$, $\alpha_0 x_0$ is also a singular point of $\sum_{n=0}^{\infty} h_n(x)$. Since $|\alpha_i|<R(X_0)-\epsilon$, $|\alpha_0|\leq R(X_0)-\epsilon$.

This contradicts that $\sum_{n=0}^{\infty} h_n(x)$ is analytic at αx_0, when $|\alpha|<R(x_0)$.

References

*) A. E. TAYLOR: (1) Analytic functions in general analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. 11 Vol. vl (1938). (2) Additions to the theory of polynomials in normed linear spaces (Tohoku M. J. 44. 1938). (3) On the properties of analytic functions in abstract spaces, Math. Ann. 115, 1938.

**) E. HILLE: Functional analysis and semi-groups. p. 85.

***) See Theorem 4. 7. 1 (HILLE, Functional analysis and semigroup, page 85). If G(hn) is non-void, then G(hn) is a c-convex c-star about θ. That is, if $X\varepsilon G(hn)$, then $\alpha X\varepsilon G(hn)$, where $|\alpha|\leqq 1$.