Some Remarks on Semi-groups and All Types of Semi-groups of Order 2, 3.

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In this paper we shall relate first to a certain necessary and sufficient condition for an algebra to be a semi-group and some properties of its principal ideals[1][0] as the addendum to our results in this Journal, vol 2, secondly to some semi-group extensions, i.e., semi-groups which contain one or two given semi-groups, and finally we shall utilize them and determine all types of semi-groups of order 2 and 3. We note that no assumption of finiteness is necessary in § 1, 2.

§ 1. The Condition and Properties of Semi-group.

It was proved in [2] that the algebra S was a semi-group if and only if it was isomorphic (anti-isomorphic) on the right (left) faithful realization system. Here we try to establish another condition which is simpler. The present notations are somewhat different from those used previously [3]. The signs $\overset{\lambda}{R}_a$, $\overset{\lambda}{L}_a$ stand for two different meanings as the case may be: one is the realization of a, i.e., the mapping[1] of S into itself, $\overset{\lambda}{R}_a(x)=x\lambda a$, $\overset{\lambda}{L}_a(x)=a\lambda x$, the other is the subset, called principal ideal. The equality as the set is written $\overset{\lambda}{R}_a \underset{i}{=} \overset{\lambda}{R}_b$ to distinguish it from the equality $\overset{\lambda}{R}_a=\overset{\lambda}{R}_b$ as the mapping. While the discussion is proceeded under an operation, the sign "λ" may be omitted.

Theorem 1. *The algebra $S(\lambda)$ is a semi-group if and only if $\overset{\lambda}{R}_a\overset{\lambda}{L}_b=\overset{\lambda}{L}_b\overset{\lambda}{R}_a$* [2] *for every $a, b \in S$.*

More generally,

Theorem 2. *Let λ and μ be semi-group operations defined in S. It holds that $\lambda \gtrsim \mu$ [4] if and only if $\overset{\mu}{R}_a\overset{\lambda}{L}_b=\overset{\lambda}{L}_b\overset{\mu}{R}_a$ for every $a, b \in S$.*

Proof. The theorems are easily obtained from the following.

$$\left\{\overset{\mu}{R}_a\overset{\lambda}{L}_b\right\}(x)=\overset{\lambda}{L}_b\left\{\overset{\mu}{R}_a(x)\right\}=\overset{\lambda}{L}_b(x\mu a)=b\lambda(x\mu a),$$

$$\left\{\overset{\lambda}{L}_b\overset{\mu}{R}_a\right\}(x)=\overset{\mu}{R}_a\left\{\overset{\lambda}{L}_b(x)\right\}=\overset{\mu}{R}_a(b\lambda x)=(b\lambda x)\mu a$$

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0) Numbers in brackets [ ] refer to the references at the end of the paper.

1) We called it a transformation previously [2].

2) We defined $\left\{\overset{\lambda}{R}_a\overset{\lambda}{L}_b\right\}(x)=\overset{\lambda}{L}_b\left\{\overset{\lambda}{R}_a(x)\right\}$ in [2]

for every $a, b, x \in S$. It becomes Theorem 1 in case that $\lambda=\mu$.

Theorem 3. *If $x \in R_a$, then $R_x \subset R_a$, and if $x \in L_a$ then $L_x \subset L_a$.*

Proof, $x=ya$ for some $y \in S$. $zx=z(ya)=(zy)a$ for any $z \in S$; hence $R_x \subset R_a$. Similarly $L_x \subset L_a$.

Let S be a semi-group with one at least idempotent hereafter.

Theorem 4. *Let a be an idempotent of S.*

(1) *If $x \in R_a$, then $xa=x$.*

(2) *If and only if $R_a \underset{i}{=} S$, a is a right unit.*

(3) *If $R_a \underset{i}{=} \{a\}$ and $ab=b$, then $R_b \underset{i}{=} \{b\}$.*

(4) *If $R_a \underset{i}{=} S$ and $ac=ab$, then $xc=xb$ for all $x \in S$.*

In the dual cases they are similar except slight modification.

Proof. (1) Since $x=ya$ for some $y \in S$, $xa=(ya)a=y(aa)=ya=x$. (2) is evident by (1). (3) $xb=x(ab)=(xa)b=ab=b$ for all x. (4) Using (2), $xc=(xa)c=x(ac)=x(ab)=(xa)b=xb$.

§ 2 Semi-group Extensions

Let A and B be disjoint semi-groups with the operations λ and μ respectively. We shall construct some sorts of semi-proups which include A and B as the sub-semi-groups keeping the operations invariant.

The set of all pairs (x, y) where $x \in A$ and $y \in B$ is called the direct product of A and B. Its operation ν is defined as $(x, y)\,\nu\,(x', y')=(x \lambda x', y \mu y')$. Then we have without difficulty

Theorem 5. *The direct product $D(\nu)$ of semi-groups $A(\lambda)$ and $B(\mu)$ is a semi-group.*

The union C of $A(\lambda)$ and $B(\mu)$ will become a semi-group, if we give such operations as seen in the below theorems, which are all proved by dint of Theorem 1. In the following theorems we don't mention that $A(\lambda)$ and $B(\mu)$ are semi-groups, $A(\lambda) \cap B(\mu)=0$ and $C(\nu)=A(\lambda) \cup B(\mu)$.

Theorem 6. *If ν is given as:*

$$x\,\nu\,y=x\,\lambda\,y \quad \text{for } x, y \in A$$
$$x\,\nu\,y=x\,\mu\,y \quad \text{for } x, y \in B$$
$$x\,\nu\,y=y\,\nu\,x=y \quad \text{for } x \in A,\ y \in B,$$

then $C(\nu)$ is a semi-group.

Before the proof we explain the notaions. By $\overset{\nu}{R}_x=\left(\overset{\lambda}{R}_x, \overset{\mu}{R}_x\right)$ we mean the mapping $\overset{\nu}{R}_x$ of C into itself by which $\overset{\nu}{R}_x(z)=\overset{\lambda}{R}_x(z)$ for $z \in A$, $\overset{\nu}{R}_x(z)=\overset{\mu}{R}_x(z)$ for $z \in B$. Especially the invariant mapping is denoted by E, and the mapping of A or B into only an element p is denoted by Z_p. We often denote $\overset{\lambda}{R}_x\overset{\lambda}{L}_y=\overset{\lambda}{L}_y\overset{\lambda}{R}_x$ by $\overset{\lambda}{R}_x \approx \overset{\lambda}{L}_y$ for short.

Proof. Since $\overset{\nu}{R}_p=(\overset{\lambda}{R}_p, E)$, $\overset{\nu}{L}_p=(\overset{\lambda}{L}_p, E)$ for $p\in A$, and $\overset{\nu}{R}_q=\left(Z_q, \overset{\mu}{R}_q\right)$, $\overset{\nu}{L}_q=\left(Z_q, \overset{\mu}{L}_q\right)$ for $q\in B$, we have immediately

$$\overset{\nu}{R}_p\overset{\nu}{L}_q=\left(\overset{\lambda}{R}_p, E\right)\left(Z_q, \overset{\mu}{L}_q\right)=\left(Z_q, \overset{\mu}{L}_q\right)=\left(Z_q, \overset{\mu}{L}_q\right)\left(\overset{\lambda}{R}_p, E\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p$$

Similarly $\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_q,\overset{\mu}{R}_q\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q$, $\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, E\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p$,

$$\overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_{q\mu q}, \overset{\mu}{R}_q\overset{\mu}{L}_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q \ .$$

Theorem 7. *Suppose that $A(\lambda)$ has a two-sided zero 0. If ν is defined as*

$$x\,\nu\,y=x\,\lambda\,y \quad \text{for } x, y\in A\ , \quad x\,\nu\,y=x\,\mu\,y \quad \text{for } x, y\in B,$$
$$x\,\nu\,y=y\,\nu\,x=0 \qquad \text{for } x\in A,\ y\in B,$$

then $C(\nu)$ is a semi-group.

Proof. Since $\overset{\nu}{R}_p=\left(R_p, Z_o\right)$, $\overset{\nu}{L}_p=\left(\overset{\lambda}{L}_p, Z_o\right)$ for $p\in A$, and $\overset{\nu}{R}_q=\left(Z_o, \overset{\mu}{R}_q\right)$, $L_q=\left(Z_o, \overset{\mu}{L}_q\right)$ for $q\in B$, we have

$$\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p, \qquad \overset{\nu}{R}_p\overset{\nu}{L}_q=\left(Z_o, Z_o\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p,$$

$$\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_o, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q, \qquad \overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_o, \overset{\mu}{R}_q\overset{\mu}{L}_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q.$$

Theorem 8 *Let $A(\lambda)$ include a two-sided zero 0 and let $B(\mu)$ be defined as $x\mu y=x$. If $C(\nu)$ is given as:*

$$x\,\nu\,y=x\,\lambda\,y \quad \text{for} \quad x, y\in A, \quad x\,\nu\,y=x\,\mu\,y \quad \text{for } x, y\in B,$$
$$x\,\nu\,y=0 \quad \text{for } x\in A,\ y\in B, \quad x\,\nu\,y=x \quad \text{for } x\in B,\ y\in A,$$

then $C(\nu)$ is a semi-group.

Proof. $\overset{\nu}{R}_p=\left(\overset{\lambda}{R}_p, E\right)$, $L_p=\left(\overset{\lambda}{L}_p, Z_o\right)$, $\overset{\nu}{R}_q=\left(Z_o, E\right)$,[3] $\overset{\nu}{L}_q=\left(Z_q, Z_q\right)$ for $p\in A$, $q\in B$.

Then
$$\overset{\nu}{R}_p\overset{\nu}{L}_p=\left(\overset{\lambda}{R}_p\overset{\lambda}{L}_p, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_p, \quad \overset{\nu}{R}_p\overset{\nu}{L}_q=\left(Z_q, Z_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_p\ ,$$
$$\overset{\nu}{R}_q\overset{\nu}{L}_p=\left(Z_o, Z_o\right)=\overset{\nu}{L}_p\overset{\nu}{R}_q, \qquad \overset{\nu}{R}_q\overset{\nu}{L}_q=\left(Z_q, Z_q\right)=\overset{\nu}{L}_q\overset{\nu}{R}_q\ .$$

Theorem 9 *Let $A(\lambda)$ be defined as $x\,\lambda\,y=y$. If ν is given as:*

$$x\,\nu\,y=r \quad (\text{fixed} \in A) \quad \text{for } x\in A,\ y\in B, \quad x\,\nu\,y=y \quad \text{for } x\in B, \quad y\in A,$$
$$x\,\nu\,y=x\,\lambda\,y \quad \text{for } x, y\in A, \quad x\,\nu\,y=x\,\mu\,y \quad \text{for } x, y\in B,$$

then $C(\nu)$ is a semi-group.

Proof. $\overset{\nu}{R}_p=\left(Z_p, Z_p\right)$, $\overset{\nu}{L}_p=\left(E, Z_r\right)$,[4] $\overset{\nu}{R}_q=\left(Z_r, \overset{\mu}{R}_q\right)$, $L_q=\left(E, \overset{\mu}{L}_q\right)$ *for $p\in A$, $q\in B$.*

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3) $\overset{\mu}{R}_q=E$, $\overset{\mu}{L}_q=Z_q$ *for $q\in B$.*

4) $\overset{\lambda}{L}_p=E$, $\overset{\lambda}{R}_p=Z_p$ *for $p\in A$.*

Then $\overset{\nu}{R}_p \overset{\nu}{L}_p = (Z_p,\ Z_p) = \overset{\nu}{L}_p \overset{\nu}{R}_p,\ \ \overset{\nu}{R}_p \overset{\nu}{L}_b = (Z_p,\ Z_p) = \overset{\nu}{L}_q \overset{\nu}{R}_q,$

$$\overset{\nu}{R}_q \overset{\nu}{L}_p = (Z_r,\ Z_r) = \overset{\nu}{L}_p \overset{\nu}{R}_q,\quad \overset{\nu}{R}_q \overset{\nu}{L}_q = (Z_r,\ \overset{\mu}{R}_q \overset{\mu}{L}_q) = \overset{\nu}{L}_q \overset{\nu}{R}_q.$$

As the special cases we consider the one-adjoined extension i. e., the semi-group A^* obtained by adjoining only an idempotent s to a semi-group A.

Corollary *Let* $C(\nu) = A(\lambda) \cup \{s\}$ *where* $s \bar{\in} A(\lambda)$. *If* ν *is given as follows,* $C(\nu)$ *is a semi-group in each each case of* (1)～(5).

(1) $x\,\nu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $\quad x\,\nu\,s = s\,\nu\,x = x$ *for* $x \in A$, $\quad s\,\nu\,s = s$.

(2) $x\,\nu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $\quad x\,\nu\,s = s\,\nu\,x = s$ *for* $x \in A$, $\quad s\,\nu\,s = s$.

(3) $A(\lambda)$ *has a two-sided zero 0.*

$x\,\nu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $\quad x\,\nu\,s = s\,\nu\,x = 0$ *for* $x \in A$,

$s\,\nu\,s = s$

(4) $A(\lambda)$ *has a two-sided zero 0.*

$x\,\nu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $\quad x\,\nu\,s = 0$,

$s\,\nu\,x = s$ *for* $x \in A$, $\quad s\,\nu\,s = s$.

(5) $A(\lambda)$ *is defined as* $x\,\lambda\,y = y$.

$x\,\nu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $\quad x\,\nu\,s = p$ (*fixed* $\in A$) *for* $x \in A$,

$s\,\nu\,x = x$ *for* $x \in A$, $\quad s\,\nu\,s = s$.

Next, as to isomorphism between the same kind of one-adjoined extensions, we have

Theorem 10. *Let* C *and* C' *be the same kind (1) or (2) of one-adjoined extensions of* A *and* A' *repectively.* C *is isomorphic with* C' *if and only if* A *is isomorphic with* A'.

Proof. Suppose C is isomorphic with C'. Let a and a' be units or zeros of C and C' respectively. Then by the uniqueness of a unit or zero we see that a is mapped to a'. Accordingly A is isomorphic with A'. The converse is clear.

Now we compose non-universal[5] one-adjoined extension of a given semi-group. Let $B(\mu) = A(\lambda) \cup \{s\}$ where $A(\lambda)$ is a semi-group and $s \bar{\in} A(\lambda)$.

Theorem 11. *If* μ *is defined as:*

$x\,\mu\,y = x\,\lambda\,y$ *for* $x,\ y \in A$, $x\,\mu\,s = x\,\lambda\,t$, $s\,\mu\,x = t\,\lambda\,x$ *for* $x \in A$, t(*fiexed*) $\in A$,

$s\,\mu\,s = t\,\lambda\,t$,

then $B(\mu)$ *is a semi-group.*

Proof. $\overset{\mu}{R}_p = (\overset{\lambda}{R}_p,\ t\lambda p)$,[6] $\overset{\mu}{L}_p = (\overset{\lambda}{L}_p,\ p\lambda t)$ *for* $p \in A$, and $\overset{\mu}{R}_s = (\overset{\lambda}{R}_t,\ t\,\lambda\,t)$, $\overset{\mu}{L}_s = (\overset{\lambda}{L}_t,\ t\,\lambda\,t)$

From them we readily have $\overset{\mu}{R}_p \approx \overset{\mu}{L}_p,\ \ \overset{\mu}{R}_p \approx \overset{\mu}{L}_s,\ \ \overset{\mu}{R}_s \approx \overset{\mu}{L}_p,\ \ \overset{\mu}{R}_s \approx \overset{\mu}{L}_s$

The following theorem is worth notice.

5) We mean by it that the one-adjoined extension $B(\mu)$ is a non-universal. See [1] with respect to "universal."

6) By $\overset{\mu}{R}_p = (\overset{\lambda}{R}_p,\ t\,\lambda\,p)$ we mean that $\overset{\mu}{R}_p(z) = \overset{\lambda}{R}_p(z)$ for $z \in A$, and $\overset{\mu}{R}_p(s) = t\,\lambda\,p$.

Theorem 12. *If $A(\lambda)$ has a two-sided unit, the non-universal one-adjoined extensions are no other than ones above shown by Theorem 11.*

Proof. Suppose that $B(\mu)$ be the non-universal one-adjoined extension of $A(\lambda)$. Let a be a two-sided unit of $A(\lambda)$, and let $s\mu a=p$, $a\mu s=q$, and $s\mu s=u$. Then $\overset{\mu}{R_a}=(E,\ p)$, $\overset{\mu}{L_a}=(E,\ q)$, moreover we set $\overset{\mu}{R_s}=(R'_s,\ u)$, $\overset{\mu}{L_s}=(L'_s,\ u)$. Since $\overset{\mu}{R_a}\approx\overset{\mu}{L_a}$ according to Theorem 1, we see that $p=q$. On the other hand it follows from [5] that $\overset{\mu}{R_s}\overset{\mu}{R_p}=\overset{\mu}{R_a}$, $\overset{\mu}{L_s}\overset{\mu}{L_t}=\overset{\mu}{L_p}$, concluding that $(R'_s,\ u)=(\overset{\lambda}{R_p},\ s\mu p)$, $(L'_s,\ u)=(\overset{\lambda}{L_p},\ p\mu s)$, consequently $R'_s=\overset{\lambda}{R_p}$, $L'_s=\overset{\lambda}{L_p}$ and $u=p\mu s=s\mu p$. We get at once $u=p\lambda p$. The proof has been completed.

§ 3 Addendum.

For the preparation of § 4, 5, a few theorems will be added.

Theorem 13. *A finite semi-group has at least an idempotent* [6].

Theorem 14. *A finite semi-group S is a right (left) groupoid*[7] *if and only if $L_x \underset{i}{=} S$ ($R_x \underset{i}{=} S$) for every $x \in S$. Especially it is a group if and only if $R_x \underset{i}{=} S$ as well as $L_x \underset{i}{=} S$ for every $x \in S$.*

Theorem 15. *If the algebra S has an idempotent a and every L_x(or R_x)is either E or Z_a for every $x \in S$, then S is a semi-group.*

Proof of Theorem 15. We see that $R_a=Z_a$. Let us consider two cases: (1) $L_a=E$, (2) $L_a=Z_a$.

(1) When $L_a=E$, it follows that $L_x=E$ for every $x \in S$. This is out of the question [8].

(2). When $L_a=Z_a$, we see that $R_x(a)=a$ for all $x \in S$ and $R_x\approx L_x$ for every $x \in S$. Hence S is a semi-group by Theorem 1.

In the next two paragraphs, we shall determine all types of semigroups, up to isomorphism, defined in $\{a,\ b\}$ and in $\{a, b, c\}$.

§ 4. Semi-groups of Order 2.

We can see easily that the following 5 operations $\lambda_1\sim\lambda_4$ and μ defined in $\{a,\ b\}$ are all semi-groups.[8]

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7) See [7].

8) We denote, for example, the table

	a	b
a	a	b
b	a	b

by

a	b
a	b

$$\underset{\lambda_1}{\begin{array}{|cc|}\hline a & b \\ a & b \\ \hline\end{array}} \qquad \underset{\lambda_2}{\begin{array}{|cc|}\hline a & a \\ a & b \\ \hline\end{array}} \qquad \underset{\lambda_3}{\begin{array}{|cc|}\hline a & b \\ b & a \\ \hline\end{array}} \qquad \underset{\lambda_4}{\begin{array}{|cc|}\hline a & a \\ b & b \\ \hline\end{array}} \qquad \underset{\mu}{\begin{array}{|cc|}\hline a & a \\ a & a \\ \hline\end{array}}$$

In fact, λ_1, λ_4 and μ are semi-groups by Theorem 15, λ_2 by (3) of the Corollary, λ_3 by Theorem 14.

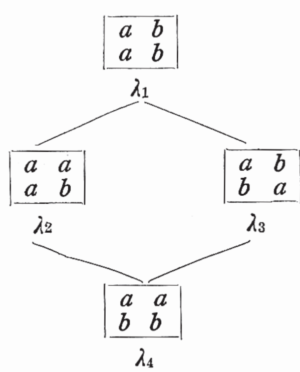
It can be proved, furthermore, that semi-groups of order 2 are nothing but these 5 types up to isomorphism. In order to prove this it is sufficient to discuss the following 3 types among all algebras which are possible to be given in $\{a, b\}$.

$$\underset{\nu_1}{\begin{array}{cc} a & b \\ b & b \end{array}} \qquad \underset{\nu_2}{\begin{array}{cc} a & b \\ a & a \end{array}} \qquad \underset{\nu_3}{\begin{array}{cc} a & a \\ b & a \end{array}}$$

Though ν_1 is isomorphic to λ_2, ν_2 is not a semi-group, neither ν_3, because $R_b \neq L_b$.

Let us now study the ordering in the universal semi-group system. By Theorem 2, we see $\lambda_2 \not\gtreqless \lambda_3$; and $\lambda_1 \underset{\not<}{\geqq} \lambda_2$, $\lambda_1 \underset{\not<}{\geqq} \lambda_3$ dually $\lambda_4 \underset{\not>}{\leqq} \lambda_2$, $\lambda_4 \underset{\not>}{\leqq} \lambda_3$.

The diagram of the universal semi-group system of order 2 is as follows:



where λ_1 is a right groupoid, λ_4 a left groupoid, λ_3 a group, λ_2 a semi-lattice.

§ 5. Semi-groups of order 3.

1. *Non-universal Semi-groups.*

Without loss of generality, it may be assumed that c does not belong to the value range[9] of $S=\{a, b, c\}$, and $\{a, b\}$ is a sub-semi-group of S; and so all the types of semi-groups $\{a, b\}$ are as follows up to isomorphism or anti- isomorphism.

$$\underset{(1)}{\begin{array}{cc} a & a \\ a & a \end{array}} \qquad \underset{(2)}{\begin{array}{cc} a & b \\ a & b \end{array}} \qquad \underset{(3)}{\begin{array}{cc} a & a \\ a & b \end{array}} \qquad \underset{(4)}{\begin{array}{cc} a & b \\ b & a \end{array}}$$

Now, we shall discuss (1)～(4) successively.

9) By the value range A^* of the subset A we mean the set composed of elements $z=xy$ for $x, y \in A$.

(1) $ac=ca=a$ follows from $R_c\approx L_a$, $L_c\approx R_a$; $bc=cb=a$ from $R_b\approx L_c$, $L_b\approx R_c$ (Theorem 1).

we have

$$\underset{\mu_1}{\begin{array}{|ccc|}\hline a & a & a \\ a & a & a \\ a & a & a \\ \hline\end{array}} \qquad \underset{\mu_2}{\begin{array}{|ccc|}\hline a & a & a \\ a & a & a \\ a & a & b \\ \hline\end{array}}$$

(2) From Theorem 1 and 4, at once $ca=a$, $cb=b$; and we get

$$\underset{\mu_3}{\begin{array}{|ccc|}\hline a & b & a \\ a & b & a \\ a & b & a \\ \hline\end{array}}$$

with which another is isomorphic.

The above μ_1, μ_2 and μ_3 prove to be semi-groups directly from Theorem 1.

(3) (4) By Theorem 11 and 12, we have

$$\underset{\mu_4}{\begin{array}{|ccc|}\hline a & a & a \\ a & b & a \\ a & a & a \\ \hline\end{array}} \qquad \underset{\mu_5}{\begin{array}{|ccc|}\hline a & a & a \\ a & b & b \\ a & b & b \\ \hline\end{array}} \qquad \underset{\mu_6}{\begin{array}{|ccc|}\hline a & b & a \\ b & a & b \\ a & b & a \\ \hline\end{array}} \qquad \underset{\mu_7}{\begin{array}{|ccc|}\hline a & b & b \\ b & a & a \\ b & a & a \\ \hline\end{array}}$$

Moreover, adding

$$\underset{\mu_8}{\begin{array}{|ccc|}\hline a & a & a \\ b & b & b \\ a & a & a \\ \hline\end{array}}$$

which is anti-isomorphic with μ_3, we have obtained all non-universal semi-groups $\mu_1 \sim \mu_8$.

2 *Universal Semi-groups.*

Without loss of generality, R_a and L_a may be assumed only as follows:

(1)	$L_a=(a, b, c)$,	$R_a=(a, b, c)$, [10]
(2)	$L_a=(a, b, c)$,	$R_a=(a, b, b)$,
(3)	$L_a=(a, b, c)$,.	$R_a=(a, b, a)$,
(4)	$L_a=(a, b, c)$,	$R_a=(a, a, a)$,
(5)	$L_a=(a, a, a)$,	$R_a=(a, b, b)$,
(6)	$L_a=(a, a, a)$,	$R_a=(a, b, a)$,
(7)	$L_a=(a, b, a)$,	$R_a=(a, b, a)$,
(8)	$L_a=(a, b, b)$,	$R_a=(a, b, b)$,
(9)	$L_a=(a, a, a)$,	$R_a=(a, a, a)$,
(10)	$L_a=(a, b, a)$,	$R_a=(a, a, c)$.

Because it is necessary that $R_a\approx L_a$; and the others are isomorphic or anti-isomorphic with one of the above by the mapping $\begin{pmatrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a & c & b \end{pmatrix}$. Now, we denote by $[b, c]$ the value range of the subset $\{b, c\}$. Successively the cases (1)$\sim$(10) will

10) By (a, b, a), for example, we mean the mapping $\begin{pmatrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a & b & a \end{pmatrix}$.

be discussed.

(1) When $a \bar{\in} [b, c]$, we have from (1) of Corollary and Theorem 10

$$\begin{array}{|ccc|} a & b & c \\ b & b & b \\ c & b & b \end{array} \qquad \begin{array}{|ccc|} a & b & c \\ b & b & c \\ c & b & c \end{array} \qquad \begin{array}{|ccc|} a & b & c \\ b & b & b \\ c & b & c \end{array} \qquad \begin{array}{|ccc|} a & b & c \\ b & b & c \\ c & c & b \end{array} \qquad \begin{array}{|ccc|} a & b & c \\ b & b & b \\ c & c & c \end{array}$$

λ_1 λ_2 λ_3 λ_4 λ_5

When $a \in [b, c]$, we can suppose $cb=a$ or $cc=a$, to which others are mapped. If $cb=a$, then by Theorem 3, we get (group)

$$\begin{array}{|ccc|} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

λ_6

If $cc=a$, we see that $cb=bc=b$ by Theorem 3, and $bb=b$ from $R_c \approx L_b$ (Theorem 1).

$$\begin{array}{|ccc|} a & b & c \\ b & b & b \\ c & b & a \end{array}$$

λ_7

(2) By Theorem 1 and 3, $a \bar{\in} [b, c]$; considering (4) of Theorem 4 and $R_b \subset R_a$ we have

$$\begin{array}{|ccc|} a & b & c \\ b & b & b \\ b & b & b \end{array} \qquad \begin{array}{|ccc|} a & b & c \\ b & b & c \\ b & b & c \end{array}$$

λ_8 λ_9

(3) By (4) of Theorem 4, $L_c=(a, b, c)$; we get $bc=b$ from $R_a \approx L_b$ and $bb=b$ from $R_c \approx L_b$, $R_b \subset R_a$.

$$\begin{array}{|ccc|} a & b & c \\ b & b & b \\ a & b & c \end{array}$$

λ_{10}

(4) See the proof of Theorem 15

$$\begin{array}{|ccc|} a & b & c \\ a & b & c \\ a & b & c \end{array}$$

λ_{11}

(5) & (6) By (3) of Theorem 4, it holds $L_b=(b, b, b)$; and so either b or c is a right-unit. Hence we have semi-groups each of which is isomorphic with one belonging to (1)$\sim$(4).

(7) From Theorem 3 follows that neither R_b nor L_b contains c; hence $cc=c$. On the other hand, $b \in R_c$, $b \in L_c$, that is, $bc=cb=b$, showing that c is a unit. Therefore this case is reduced to the previous one.

(8) Similarly $cc=c$. From this it concludes that $bc=cb=b$, because we require $R_a \approx L_c$, $R_c \approx L_a$. If $bb=a$, then $R_a \not\approx L_b$. Hence $bb=b$; we have

$$\begin{array}{|ccc|} a & b & b \\ b & b & b \\ b & b & c \end{array}$$

λ_{12}

(9) Let us investigate the case that a semi-group has no idempotent but a and the value range $[b, c]$ contains a. For, if $a \bar{\in} [b, c]$, then we have from (2) of Corollary and Theorem 10

$$
\begin{array}{ccc} a & a & a \\ a & b & c \\ a & b & c \end{array}
\qquad
\begin{array}{ccc} a & a & a \\ a & b & b \\ a & b & c \end{array}
\qquad
\begin{array}{ccc} a & a & a \\ a & b & c \\ a & c & b \end{array}
\qquad
\begin{array}{|ccc|} \hline a & a & a \\ a & b & b \\ a & c & c \\ \hline \end{array}
$$

$$\lambda'_{10} \qquad \lambda'_3 \qquad \lambda'_7 \qquad \lambda_{13}$$

whereas λ'_{10} is isomorphic with λ_{10}, λ'_3 with λ_3, λ'_7 with λ_7; and λ_{13} is anti-isomorphic with λ_{10}.

If one at least of b and c is idempotent, then the semi-group is isomorphic with one of (1)$\sim$(8). Now we take up only the following cases, which are all out of our consideration.

$$
\begin{array}{ccc} a & a & a \\ a & a & \\ a & & a \end{array}
\qquad
\begin{array}{ccc} a & a & a \\ a & a & \\ a & & b \end{array}
\qquad
\begin{array}{ccc} a & a & a \\ a & c & \\ a & & b \end{array}
$$

$$\text{i)} \qquad \text{ii)} \qquad \text{iii)}$$

i) The element cb must be either b or c, but whatever cb is, $R_b \not\approx L_c$.

ii) Either cb or bc must be c. Then $R_b \not\approx L_c$ or $L_b \not\approx R_c$.

iii) By the assumption, $cb = a$ or $bc = a$. However it follows that $R_b \not\approx L_c$ or $L_b \not\approx R_c$.

10) It follows from $L_a \approx R_b$ that $cb = b$, contradicting to $R_a \approx L_c$. Hence there is none with $L_a = (a, b, a)$, $R_a = (a, a, c)$.

In addition to $\lambda_1 \sim \lambda_{13}$, we have the remaining ones which are anti-isomorphic with the former.

$$
\begin{array}{|ccc|} \hline a & a & a \\ b & b & b \\ c & c & c \\ \hline \end{array}
\qquad
\begin{array}{|ccc|} \hline a & b & b \\ b & b & b \\ c & b & b \\ \hline \end{array}
\qquad
\begin{array}{|ccc|} \hline a & b & b \\ b & b & b \\ c & c & c \\ \hline \end{array}
$$

$$\lambda_{14} \qquad \lambda_{15} \qquad \lambda_{16}$$

We can easily see that $\lambda_1 \sim \lambda_{16}$ thus obtained are semi-groups which are not isomorphic each other.

3 *The Ordering of the Universal Semi-group System*

At first we define a term as following. If the system $\mathfrak{N}$ of universal semi-group operations defined in a set S satisfies the conditions (1) and (2) as follows, $\mathfrak{N}$ is called the normal represent system of universal semi-groups with respect to S.

(1) For any $\lambda, \mu \in \mathfrak{N}$ $(\lambda \neq \mu)$, one is not isomorphic with the other.

(2) For any $\lambda \in \mathfrak{N}$, $\mathfrak{N}$ contains ν which is identically anti-isomorphic with λ, that is, $\nu = \lambda^{\mathfrak{e}}$ [10] where $\mathfrak{e}$ is an identical translation on S.

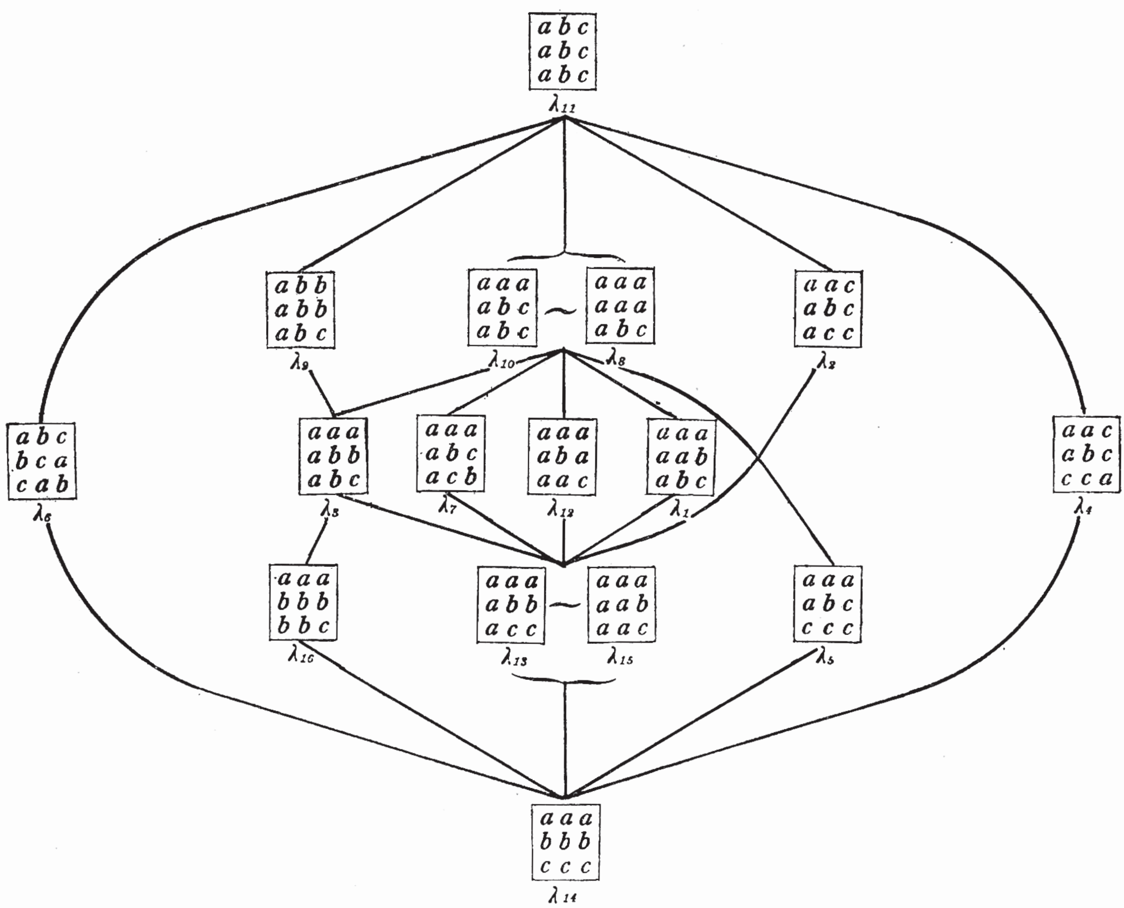
For example, we have as the normal represent system of universal semi-groups:

$$\underset{\lambda_1}{\begin{vmatrix} a & a & a \\ a & a & b \\ a & b & c \end{vmatrix}} \quad \underset{\lambda_2}{\begin{vmatrix} a & a & c \\ a & b & c \\ a & c & c \end{vmatrix}} \quad \underset{\lambda_3}{\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}} \quad \underset{\lambda_4}{\begin{vmatrix} a & a & c \\ a & b & c \\ c & c & a \end{vmatrix}} \quad \underset{\lambda_5}{\begin{vmatrix} a & a & a \\ a & b & c \\ c & c & c \end{vmatrix}} \quad \underset{\lambda_6}{\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}} \quad \underset{\lambda_7}{\begin{vmatrix} a & a & a \\ a & b & c \\ a & c & b \end{vmatrix}} \quad \underset{\lambda_8}{\begin{vmatrix} a & a & a \\ a & a & a \\ a & b & c \end{vmatrix}}$$

$$\underset{\lambda_9}{\begin{vmatrix} a & b & b \\ a & b & b \\ a & b & c \end{vmatrix}} \quad \underset{\lambda_{10}}{\begin{vmatrix} a & a & a \\ a & b & c \\ a & b & c \end{vmatrix}} \quad \underset{\lambda_{11}}{\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix}} \quad \underset{\lambda_{12}}{\begin{vmatrix} a & a & a \\ a & b & a \\ a & a & c \end{vmatrix}} \quad \underset{\lambda_{13}}{\begin{vmatrix} a & a & a \\ a & b & b \\ a & c & c \end{vmatrix}} \quad \underset{\lambda_{14}}{\begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}} \quad \underset{\lambda_{15}}{\begin{vmatrix} a & a & a \\ a & a & b \\ a & a & c \end{vmatrix}} \quad \underset{\lambda_{16}}{\begin{vmatrix} a & a & a \\ b & b & b \\ b & b & c \end{vmatrix}}$$

where these λ_i are isomorphic with the previously written λ_i,

The diagram of the system is obtained by Theorem 2 or [9].



As easily seen, this system forms a lattice, but the lattice depends on the represent system.

References.

[1] As to the definition of a principal ideal, see T. TAMURA, *Characterization of groupoids and semilattices by ideals in a semigroup*, Journal of Science of the Gakugei Faculty, Tokushi-

ma University. Vol. 1, (1950) p. 37.

〔2〕 T. TAMURA, *On the system of semigroup operations defined in a set*, Journal of the Gakugei Faculty, Tokushima Unlversity, (natural science) Vol. 2, (1952) pp. 2-3.

〔3〕 The paper 〔2〕 pp. 2-6.

〔4〕 The paper 〔2〕 p. 3.

〔5〕 The paper 〔2〕 p. 2

〔6〕 D. REES, *On semi-groups*, Proceedings of the Cambridge Philosophical Society, Vol. 36, pt. 4 (1940) § 1. 2.

〔7〕 The paper 〔1〕 pp. 38-39.

〔8〕 The paper 〔2〕 p. 4.

〔9〕 See 〔2〕.

〔10〕 The paper 〔2〕 p. 8.

The operation λ^{e} is defined as: $x \lambda^{e} y = y \lambda x$.

Addendum.

It is regret that I can not refer to the papers by Clifford, Suschkewitsch, etc., for lack of literature in our university. I fear that some of our results may have been contained in a study by someone.