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On the System of Semigroup Operations Defined in a Set.

By

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§ 1. Introdution.

The object of this paper is the semigroup operation system $\mathfrak{M}$ of a set E, i. e., the aggregate of all possible semigroup operations λ, μ, $\cdots$ defined in abstract set E. More strictly,

Definition 1. $\mathfrak{M}$ is the set of all λ satisfying the below conditions :

(1) To each pair of elements a and $b \in E$ corresponds a unique element $a \lambda b \in E$.

(2) λ is associative : $(a\lambda b)\lambda c = a\lambda(b\lambda c)$ for any $a, b, c \in E$.

The equality of elements of $\mathfrak{M}$ is defined as follows.

Definition 2. Two cperations λ and μ are said to de equal i. e $\lambda = \mu$, if $x\lambda y = x\mu y$ for any $x, y \in E$.

In the present paper we shall discuss how the semigroup opertaion system is ordered, and how we realize the ordering in the transformation semigroups, but there remain many problems unsolved. In order to introduce some quasi-ordering into the system we will restrict ourselves to the universal semigroup operation system $\mathfrak{U}$ of E.

Definition 3. A semigroup operation λ defined in E is called univesal if for any $c \in E$ there exist a and $b \in E$ such that $a \lambda b = c$.

By the universal semigroup operation system $\mathfrak{U}$ of E is meant the set of all universal semigroup operations defined in E.

§ 2. The Necessary and Sufficient Condition of a Semigroup.

As the preliminaries we shall relate the necessary and sufficient condition [1] that the associative law is fulfilled by an algebraic system $E(\lambda)$[1] by which is meant an abstract set with a binary operation λ. If F is a subset of the algebraic system E and $a \lambda b \in F$ whenever a and $b \in F$, we call F an algebraic subsystem of E. Although it is needless to say,

Lemma 1. An algebraic subsystem F of a semigroup E is a subsemigroup of E.

Lemma 2. If a semigroup E is homomorphic or anti-homomorphic on an algebraic system E', then E' is a semigroup.

1) we denote by $E(\lambda)$ the algebraic system E with λ when λ need to be specially assigned, but simply by E when there is no fear of confusion.

Definition 4. A single-valued mapping T of an abstract set M into itself is called a transformation of M, i. e. , to any $x \in M$ corresponds a unique element $Tx \in M$. Of course we define equality of two transformations as

$$T = S \quad \text{if} \quad Tx = Sx \quad \text{for all} \quad x \in M.$$

If the product $R = TS$ of transformations T and S is given as $Rx = S(Tx)$ for $x \in M$, then the set of all transformations of M obviously forms a semigroup [2] , whence the set is called the transformation semigroup $\mathfrak{T}_M$ on M, and a subsemigroup of $\mathfrak{T}_M$ is called a transformation subsemigroup on M.

As the special transformation system, we define a realization system and a faithful realization system as following.

Definition 5. Let a be an element of the algebraic system E with an operation λ. Then the transformation $R_\lambda(a)$ given as $R_\lambda(a)x = x\lambda a$ (for $x \in E$) is called the right λ—realization of a in E, the transformation $L_\lambda(a)$ given as $L_\lambda(a)x = a\lambda x$ (for $x \in E$) is called the left λ—realization of a in E.

Letting $\mathfrak{R}_\lambda = [R_\lambda(a) \mid a \in E]$, $\mathfrak{L}_\lambda = [L_\lambda(a) \mid a \in E]$, $\mathfrak{R}_\lambda$ (or $\mathfrak{L}_\lambda$) is called the right (left) λ—realization system of $E(\lambda)$, affording little convenience to our general discussion [3].

Definition 6. Let $\bar{E}(\bar{\lambda})$ be the extended algebraic system of $E(\lambda)$, which is obtained by abjoining only one new element p to $E(\lambda)$ and defining the operation $\bar{\lambda}$ in $\bar{E}$ as follows.

$$x\,\bar{\lambda}\,y = x\,\lambda\,y \qquad \text{if} \quad x, y \in E,$$
$$p\,\bar{\lambda}\,x = x\,\bar{\lambda}\,p = x \qquad \text{if} \quad x \in \bar{E}.$$

As easily shown, $\bar{E}$ is a semigroup if and only if E is a semigroup [4].

Definition 7. Let $a \in E(\lambda) \subset \bar{E}(\bar{\lambda})$. The right (left) λ— realization of a in $\bar{E}(\bar{\lambda})$ is called right (left) faithful λ—realization of a, written $\bar{R}_\lambda(a)$ ($\bar{L}_\lambda(a)$); and the set of them i. e., $\bar{\mathfrak{R}}_\lambda = [\bar{R}_\lambda(a) \mid a \in E]$ or $\bar{\mathfrak{L}}_\lambda = [\bar{L}_\lambda(a) \mid a \in E]$ is called the right or left faithful λ—realization system of E respectively, where $a \leftrightarrow \bar{R}_\lambda(a)$ or $a \leftrightarrow \bar{L}_\lambda(a)$ is one-to-one.

Now we have the following theorems.

Theorem 1. *An algebraic system $E(\lambda)$ is a semigroup if and only if $\bar{R}_\lambda(a)\bar{R}_\lambda(b) = \bar{R}_\lambda(a\lambda b)$ for every $a, b \in E$.*

Theorem 1'. *An algebraic system $E(\lambda)$ is a semigroup if and only if $\bar{L}_\lambda(a)\bar{L}_\lambda(b) = \bar{L}_\lambda(b\lambda a)$ for every $a, b \in E$.*

Remark. The formula shows that $\bar{\mathfrak{R}}_\lambda$ ($\bar{\mathfrak{L}}_\lambda$) is an algebraic system and $E(\lambda)$ is isomorphic (anti-isomorphic) on $\bar{\mathfrak{R}}_\lambda$ ($\bar{\mathfrak{L}}_\lambda$).

Proof of Theorem 1. Suppose that $E(\lambda)$ is a semigroup. By the assumption of λ and the definition of $\bar{R}_\lambda$,

$$\left\{\bar{R}_\lambda(a)\bar{R}_\lambda(b)\right\}x=\bar{R}_\lambda(b)\left\{\bar{R}_\lambda(a)x\right\}=\bar{R}_\lambda(b)(x\,\lambda\,a)=(x\,\lambda\,a)\,\lambda\,b$$

$$=x\,\lambda(a\lambda b)=\bar{R}_\lambda(a\,\lambda\,b)\,x \qquad \text{for} \quad x\in E,$$

and $\left\{\bar{R}_\lambda(a)\,\bar{R}_\lambda(b)\right\}p=\bar{R}_\lambda(b)\left\{\bar{R}_\lambda(a)\,p\right\}=\bar{R}_\lambda(b)\,a=a\,\lambda\,b=\bar{R}_\lambda(a\,\lambda\,b)\,p.$

In short, $\left\{\bar{R}_\lambda(a)\,\bar{R}_\lambda(b)\right\}x=\bar{R}_\lambda(a\,\lambda\,b)\,x$ for any $x\in\bar{E}$.

Finally we have $\bar{R}_\lambda(a)\,\bar{R}_\lambda(b)=\bar{R}_\lambda(a\,\lambda\,b).$ (1)

Conversely suppose (1). It follows from (1) that $\bar{\mathfrak{R}}_\lambda$ is an algebraic subsystem of the semigroup $\mathfrak{T}_{\bar{E}}$ and thet $\bar{\mathfrak{R}}_\lambda$ is isomorphic on $E(\lambda)$ under the mapping: $\bar{R}_\lambda(x)\leftrightarrow x$. Hence $E(\lambda)$ is immediately concluded to be a semighoup by means of Lemma 1 and 2.

We can similarly prove Theorem 1′. If the correspondence between $E(\lambda)$ and its realization system $\mathfrak{R}_\lambda(\mathfrak{L}_\lambda)$ is one-to-one , $\mathfrak{R}_\lambda(\mathfrak{L}_\lambda)$ is isomorphic with the faithful realization system $\bar{\mathfrak{R}}_\lambda(\bar{\mathfrak{L}}_\lambda)$. Therefore we have.

Corollary 1. *Assum that $a\leftrightarrow R_\lambda(a)$ is one-to-one. In order that $E(\lambda)$ is a semigroup, it is necessary and sufficient that $R_\lambda(a)\,R_\lambda(b)=R_\lambda(a\,\lambda\,b)$ for every a , $b\in E$.*

Corollary 2. *Assume that $a\leftrightarrow L_\lambda(a)$ is one-to-one. In ordey that $E(\lambda)$ is a semigroup, it is necessary and sufficient that $L_\lambda(a)\,L_\lambda(b)=L_\lambda(b\,\lambda\,a)$ for every a , $b\in E$.*

§ 3. The Ordering in $\mathfrak{U}$.

Definition 8. If $(a\,\lambda\,b)\,\mu\,c=a\,\lambda\,(b\,\mu\,c)$ for any a , b , $c\in$ E, then we denote it by $\lambda\gtrsim\mu$, or by $\mu\lesssim\lambda$.

Theorem 2. *Let λ, $\mu\in\mathfrak{M}$ and $\mu\in\mathfrak{U}$. If $\lambda\gtrsim\mu$ and $\mu\gtrsim\nu$, then $\lambda\gtrsim\nu$.*

Proof. For any a, b and $c\in E$,

$$\begin{aligned}(a\,\lambda\,b)\,\nu\,c&=\left\{a\,\lambda\,(b'\mu\,b'')\right\}\nu\,c && (\because\ \mu\in\mathfrak{U},\ b=b'\mu\,b'')\\ &=\left\{(a\,\lambda\,b')\,\mu\,b''\right\}\nu\,c && (\because\ \lambda\gtrsim\mu)\\ &=(a\,\lambda\,b')\,\mu\,(b''\nu\,c) && (\because\ \mu\gtrsim\nu)\\ &=a\,\lambda\left\{b'\mu(b''\nu c)\right\} && (\because\ \lambda\gtrsim\mu)\\ &=a\,\lambda\left\{(b'\mu\,b'')\nu c\right\} && (\because\ \mu\gtrsim\nu)\\ &=a\,\lambda\,(b\,\nu\,c).\end{aligned}$$

Moreover it always holds that $\lambda \gtrsim \lambda$ for every $\lambda \in \mathfrak{M}$. If we are confined to the universal semigroup operation system $\mathfrak{U}$, the relation $\gtrsim$ is a quasi-ordering [5] in $\mathfrak{U}$. Let us identify λ and μ, denote $\lambda \sim \mu$, when $\lambda \gtrsim \mu$ as well as $\lambda \lesssim \mu$. Then $\mathfrak{U}$ becomes a partially ordered set under the identification [6].

Remark. The universality of μ in Theorem 2 has an effect on the transitive law. More precisely, if it were not for the universality, the law would not necessarily hold [7]. Let us take for example the finite set M (of three elements a, b, and c) in which the three semigroup operations λ, μ[2] and ν are given as the below product tables show.

λ

left \ right	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

μ

left \ right	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

ν

left \ right	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

1° At first we must show that λ, μ, and ν are all semigroup operations. In fact, respecting μ, it is evident; as far as λ, μ are concerned, we can prove them easily dy Theorem 1 or dy direct method [8].

2° $\lambda \gtrsim \mu$

For, x, y, and z symbolling one of a, b, and c,

$(x \lambda y) \mu z = a$, $x \lambda (y \mu z) = x \lambda a = a$. Hence $(x \lambda y) \mu z = x \lambda (y \mu z)$.

3° $\mu \gtrsim \nu$

For, $(x \mu y) \nu z = a \nu z = a$, $x \mu (y \nu z) = a$, Hence $(x \mu y) \nu z = x \mu (y \nu z)$.

4° On the other hand [9] $\lambda \not\gtrsim \nu$.

For, $(b \lambda c) \nu b = b \nu b = b$, $b \lambda (c \nu b) = b \lambda a = a$. Therefore $(b \lambda c) \nu b \neq b \lambda (c \nu b)$.

Now let us define α, β as following:

$$x \alpha y = y \quad , \quad x \beta y = x \qquad \text{for every } x, \ y \in E,$$

where α, β is easily shown to belong to $\mathfrak{U}$. Then we have

Corollary 3. *$\alpha \gtrsim \lambda$ and $\lambda \gtrsim \beta$ for every $\lambda \in \mathfrak{M}$.*

Proof. For any x, y, $z \in E$,

$$(x \alpha y) \lambda z = y \lambda z, \qquad x \alpha (y \lambda z) = y \lambda z,$$

and

$$(x \lambda y) \beta z = x \lambda y, \qquad x \lambda (y \beta z) = x \lambda y \quad ;$$

---

2) μ is not univeisal.

hence $(x\,\alpha\,y)\,\lambda\,z = x\,\alpha\,(y\,\lambda\,z), \quad (x\,\lambda\,y)\,\beta\,z = x\,\lambda\,(y\,\beta\,z).$

Consequently we can assert that $\mathfrak{U}$ *is the above and below bounded partially ordered set under the mentioned adequate identification.*

§ 4. The Problem of Ordering in the Realization System.

As the validity of associative law with respect to one operation has been reduced to the problem in the faithfull realization system (cf. § 2, Theorem 1, 1′), so the comparability[3] between different operations defined in E will be considered as that between different realization systems of E.

Theorem 3. In order that $\lambda \gtrsim \mu$ for $\lambda, \mu \in \mathfrak{M}$, it is necessary and sufficient that

(1) $\bar{R}_\lambda(a)\,\bar{R}_\mu(b) = \bar{R}_\lambda(a\,\mu\,b)$ *for every* a *and* $b \in E$.

Proof. Suppose that $\lambda \gtrsim \mu$. If $x \in E \subset \bar{E}$,

$$\left\{\bar{R}_\lambda(a)\,\bar{R}_\mu(b)\right\}x = \bar{R}_\mu(b)\left\{\bar{R}_\lambda(a)\,x\right\} = \bar{R}_\mu(b)\,(x\,\lambda\,a)$$

$$= (x\,\lambda\,a)\,\mu\,b = x\,\lambda\,(a\,\mu\,b) = \bar{R}_\lambda(a\,\mu\,b)\,x\ ;$$

otherwise, $\left\{\bar{R}_\lambda(a)\,\bar{R}_\mu(b)\right\}p = \bar{R}_\mu(b)\left\{\bar{R}_\lambda(a)\,p\right\} = \bar{R}_\mu(b)\,a = a\,\mu\,b = \bar{R}_\lambda(a\,\mu\,b)\,p,$

after all $\left\{\bar{R}_\lambda(a)\,\bar{R}_\mu(b)\right\}x = \bar{R}_\lambda(a\,\mu\,b)\,x$ for any $x \in \bar{E}$.

Therefore we get $\bar{R}_\lambda(a)\,\bar{R}_\mu(b) = \bar{R}_\lambda(a\,\mu\,b)$.

Conversely if (1) holds, then we shall arrive at

$$\bar{R}_\lambda\left\{(a\,\lambda\,b)\,\mu\,c\right\} = \bar{R}_\lambda\left\{a\,\lambda\,(b\,\mu\,c)\right\}.$$

For every $a, b,$ and $c \in E$,

$$\bar{R}_\lambda\left\{a\,\lambda\,(b\,\mu\,c)\right\} = \bar{R}_\lambda(a)\,\bar{R}_\lambda(b\,\mu\,c) \qquad \text{(by Tteorem 1)}$$

$$= \bar{R}_\lambda(a)\left\{\bar{R}_\lambda(b)\,\bar{R}_\mu(c)\right\} \qquad \text{(by (1))}$$

$$= \left\{\bar{R}_\lambda(a)\,\bar{R}_\lambda(b)\right\}\bar{R}_\mu(c) \qquad \text{(by the associative law in } \mathfrak{T}_{\bar{E}}\text{)}$$

$$= \bar{R}_\lambda(a\,\lambda\,b)\,\bar{R}_\mu(c) \qquad \text{(by Theorem 1)}$$

$$= \bar{R}_\lambda\left\{(a\,\lambda\,b)\,\mu\,c\right\}. \qquad \text{(by (1))}$$

Since the correspondence $\bar{R}_\lambda(x) \to x$ is one to one, we have

$$(a\,\lambda\,b)\,\mu\,c = a\,\lambda\,(b\,\mu\,c).$$

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3) Two operations λ and μ are said to be comparable if either $\lambda \gtrsim \mu$ or $\lambda \lesssim \mu$; and are said to be incomparable if neither $\lambda \gtrsim \mu$ nor $\lambda \lesssim \mu$, denoted $\lambda \not\gtrless \mu$.

Thus the proof of this theorem has been completed.

Similarly we get

Theorem 3'. In order that $\lambda \gtrsim \mu$ for $\lambda, \mu \in \mathfrak{M}$, it is necessary and sufficient that

(1′) $\bar{L}_\mu(a)\bar{L}_\lambda(b)=\bar{L}_\mu(b\,\lambda\,a)$ for every a and $b \in E$.

we note that the above theorems need no assumption of universality and that they are the extensions of Theorem 1 and 1′. In order to establish Theorem 4 and 4′ equivalent to Theorem 3 and 3′, a few definitions have to be prepared.

Definition 9. The one-to-one correspondence $\bar{R}_\lambda(a) \leftrightarrow \bar{R}_\mu(a)$ between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$ is called the natural correspondence between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$. The natural correspondence between $\bar{\mathfrak{L}}_\lambda$ and $\bar{\mathfrak{L}}_\mu$ is also similarly defined.

Definition 10. Let A and B be twe subsets of a set and Φ be the system composed of transformations φ of $A \cup B$ into itself such that $\varphi(A) \subset A$ and $\varphi(B) \subset B$. If besides Φ there is a one-to-one correspondence f between A and B, and if f is preserved by Φ-transformations, i. e. $A \ni a \overset{f}{\leftrightarrow} b \in B$ implies $A \ni \varphi(a) \overset{f}{\leftrightarrow} \varphi(b) \in B$, then the correspondence f is said to be invarient by Φ.

Now it follows from Theorem 3 (3′) that $\bar{R}_\lambda(a)\bar{R}_\mu(b)$ (or $\bar{L}_\mu(a)\bar{L}_\lambda(b)$) is thought as the image of $\bar{R}_\lambda(a)$ ($\bar{L}_\mu(a)$) under the transformation meaning multiplication of $\bar{R}_\lambda(a)$ ($\bar{L}_\mu(a)$) by $\bar{R}_\mu(b) \in \bar{\mathfrak{R}}_\mu$ ($\bar{L}_\lambda(b) \in \bar{\mathfrak{L}}_\lambda$) in the right side.

We shall call it $\bar{\mathfrak{R}}_\mu$-transformations ($\bar{\mathfrak{L}}_\lambda$—transformations), which, of course, may also be applied to $\bar{R}_\mu(a)$ ($\bar{L}_\lambda(a)$).

Theorem 4. *Let $\lambda, \mu \in \mathfrak{M}$. In order that $\lambda \gtrsim \mu$, it is necessary and sufficient that*

(1) $\bar{\mathfrak{R}}_\lambda \bar{\mathfrak{R}}_\mu \subset \bar{\mathfrak{R}}_\lambda$,

(2) *the natural correspondence between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$ is invariant by $\bar{\mathfrak{R}}_\mu$-transformations.*

Proof. Let us suppose (1) and (2). It follows from (2) that the natural correspondence $\bar{R}_\lambda(a) \leftrightarrow \bar{R}_\mu(a)$ implies $\bar{R}_\lambda(a)\bar{R}_\mu(b) \leftrightarrow \bar{R}_\mu(a)\bar{R}_\mu(b)$ for every a and $b \in E$. On the other hand, there is an element $c \in E$ such that $\bar{R}_\lambda(a)\bar{R}_\mu(b)=\bar{R}_\lambda(c)$ by (1); and Theorem 1 shows $\bar{R}_\mu(a)\bar{R}_\mu(b)=\bar{R}_\mu(a\,\mu\,b)$. Hence we have $\bar{R}_\lambda(c) \leftrightarrow \bar{R}_\mu(a\,\mu\,b)$ concluding $c=a\,\mu\,b$ due to the definition of the natural correspondence. Thus we have arrived at the formula of Theorem 3. Conversely (1) and (2) follow immediately from Theorem 3.

Similarly we have

Theorem 4'. *In order that $\lambda \gtrsim \mu$ it is necessary and sufficient that*

(1′) $\bar{\mathfrak{L}}_\mu \bar{\mathfrak{L}}_\lambda \subset \bar{\mathfrak{L}}_\mu$, (2′) *the natural correspondence between* $\mathfrak{L}_\lambda$ *and* $\mathfrak{L}_\mu$ *is invariant by* $\bar{\mathfrak{L}}_\lambda$*- transformations.*

§ 5. Translations of Operations.

By a translation on M we mean a one-to-one transformation of M onto itself. The set of all translations of M forms a group, which is called the translation group on M, and a subgroup of which is called a translation subgroup on M. Let us denote by $\mathfrak{G}$ a translation subgroup on initially given E, and individual translation by $\mathfrak{x}$, $\mathfrak{h}$, ··etc. Then corresponding to $\mathfrak{G}$ the translation subgroup $\bar{\mathfrak{G}}$ will be defined.

Definition 11. We let a transformation $\bar{\mathfrak{x}}$ of $\mathfrak{M}$ correspond to $\mathfrak{x} \in \mathfrak{G}$ as follows:

$$\lambda \xrightarrow{\bar{\mathfrak{x}}} \lambda^{\bar{\mathfrak{x}}} \qquad \text{(for any } \lambda \in \mathfrak{M})$$

where the operation $\lambda^{\bar{\mathfrak{x}}}$ is defined as

$$a\ \lambda^{\bar{\mathfrak{x}}}\ b = (a^{\mathfrak{x}}\ \lambda\ b^{\mathfrak{x}})^{\mathfrak{x}^{-1}} \text{ [4]} \qquad \text{for any } a, b \in E.$$

Lemma 3. $(a\ \lambda^{\bar{\mathfrak{x}}}\ b)\ \lambda^{\bar{\mathfrak{x}}}\ c = a\ \lambda^{\bar{\mathfrak{x}}}\ (b\ \lambda^{\bar{\mathfrak{x}}}\ c)$.

Proof.

$$(a\ \lambda^{\bar{\mathfrak{x}}}\ b)\ \lambda^{\bar{\mathfrak{x}}}\ c = \left\{(a\ \lambda^{\bar{\mathfrak{x}}}\ b)^{\mathfrak{x}}\ \lambda\ c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{(a^{\mathfrak{x}}\ \lambda\ b^{\mathfrak{x}})\ \lambda\ c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}},$$

$$a\ \lambda^{\bar{\mathfrak{x}}}\ (b\ \lambda^{\bar{\mathfrak{x}}}\ c) = \left\{a^{\mathfrak{x}}\ \lambda\ (b\ \lambda^{\bar{\mathfrak{x}}}\ c)^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{a^{\mathfrak{x}}\ \lambda\ (b^{\mathfrak{x}}\ \lambda\ c^{\mathfrak{x}})\right\}^{\mathfrak{x}^{-1}}.$$

Utilizing that λ is associative, the given formula is proved.

Lemma 4. $\bar{\mathfrak{x}}\,\bar{\mathfrak{h}} = \overline{\mathfrak{h}\,\mathfrak{x}}$. [5]

Proof.

$$a\ \lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{h}}}\ b = (a^{\mathfrak{h}}\ \lambda^{\bar{\mathfrak{x}}}\ b^{\mathfrak{h}})^{\mathfrak{h}^{-1}} = (a^{\mathfrak{h}\mathfrak{x}}\ \lambda\ b^{\mathfrak{h}\mathfrak{x}})^{\mathfrak{x}^{-1}\mathfrak{h}^{-1}} = a\ \lambda^{\overline{\mathfrak{h}\mathfrak{x}}}\ b \qquad \text{for every } a, b \in E.$$

Hence $\lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{h}}} = \lambda^{\overline{\mathfrak{h}\mathfrak{x}}}$ for every $\lambda \in \mathfrak{M}$.

It follows from Lemma 3 and 4 that the operation $\lambda^{\bar{\mathfrak{x}}}$ belongs to $\mathfrak{M}$ and that any λ has $\lambda^{\overline{\mathfrak{x}^{-1}}}$ as its inverse image under the transformation $\bar{\mathfrak{x}}$. Thus $\bar{\mathfrak{x}}$ for $\mathfrak{x} \in \mathfrak{G}$ has been asserted to be a translation of $\mathfrak{M}$; moreover $\bar{\mathfrak{x}}$ becomes, in f ct, a translation of $\mathfrak{U}$. It is for this reason that the following lemma shows.

Lemma 5. *If* λ *is uuiversal,* $\lambda^{\bar{\mathfrak{x}}}$ *is uuiversal.*

Proof. Given any $c \in E$, we denote $c^{\mathfrak{x}}$ by c'. Since λ is universal, there exist a' and b' such that $c' = a'\ \lambda\ b'$. Letting $a = a'^{\mathfrak{x}^{-1}}$, $b = b'^{\mathfrak{x}^{-1}}$, we get $c = a\ \lambda^{\bar{\mathfrak{x}}}\ b$; thus $\lambda^{\bar{\mathfrak{x}}}$ is universal.

If the set of all $\bar{\mathfrak{x}}$ for $\mathfrak{x} \in \mathfrak{G}$ is denoted by $\bar{\mathfrak{G}}$, we have

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4) $a^{\mathfrak{x}}$ represents the image of a under the translation $\mathfrak{x}$ of E.

5) $\lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{h}}} = (\lambda^{\bar{\mathfrak{x}}})^{\bar{\mathfrak{h}}}$, $a^{\mathfrak{x}\mathfrak{h}} = (a^{\mathfrak{x}})^{\mathfrak{h}}$.

Theorem 5. $\mathfrak{G}$ *is anti-homomorphic on* $\bar{\mathfrak{G}}$. *Accordingly* $\bar{\mathfrak{G}}$ *forms a group.*

We call $\bar{\mathfrak{G}}$ a principal translation subgroup on $\mathfrak{M}$ (or $\mathfrak{U}$) to $\mathfrak{G}$. What condition does $\mathfrak{G}$ require in order that it is anti-isomorphic on $\bar{\mathfrak{G}}$? Let $\mathfrak{A}(\lambda)$ be the group of all automorphisms in an algebraic system E with λ.

Lemma 6. $\lambda = \lambda^{\bar{\mathfrak{x}}}$ *for* $\lambda \in \mathfrak{M}$ *if and only if* $\mathfrak{x} \in \mathfrak{A}(\lambda) \cap \mathfrak{G}$.

Theorem 6. $\mathfrak{G}$ *is anti-isomorphic on* $\bar{\mathfrak{G}}$ *if and only if*

$$\bigcap_{\lambda \in \mathfrak{M}} \mathfrak{A}(\lambda) \cap \mathfrak{G} = \{\mathfrak{e}\} \quad \text{[6]} \qquad \text{(cf. [10])}$$

Now we define a translation subgroup $\mathfrak{P}$ other than $\bar{\mathfrak{G}}$. $\mathfrak{P}$ shall be generated by the only one translation $\mathfrak{p}$ of $\mathfrak{M}$ or $\mathfrak{U}$, where $\mathfrak{p}$ maps any λ to $\lambda^{\mathfrak{p}}$ [7] given as

$$x \, \lambda^{\mathfrak{p}} \, y = y \, \lambda \, x \qquad \text{for } x, \ y \in E.$$

The subgroup generated by $\bar{\mathfrak{G}}$ and $\mathfrak{P}$ is called the fundamental translation subgroup of $\mathfrak{M}$ (or $\mathfrak{U}$) to $\mathfrak{G}$.

Letting $\underline{\mathfrak{x}} = \bar{\mathfrak{x}} \mathfrak{p} = \mathfrak{p} \bar{\mathfrak{x}}$, we immediately have

Corollary 4. $\bar{\mathfrak{x}} \bar{\mathfrak{y}} = \underline{\mathfrak{x}} \underline{\mathfrak{y}} = \overline{\mathfrak{y}\mathfrak{x}}$, $\quad \underline{\mathfrak{x}} \bar{\mathfrak{y}} = \bar{\mathfrak{x}} \underline{\mathfrak{y}} = \underline{\mathfrak{y}\mathfrak{x}}$.

On the relations between translations of operations and the initial set E.

Corollary 5. $E(\lambda)$ is isomorphic on $E(\lambda^{\bar{\mathfrak{x}}})$, and $E(\lambda)$ is anti-isomorphic on $E(\lambda^{\underline{\mathfrak{x}}})$.

§ 6. Relations between the Ordering and Translations of Operations.

In this paragraph the comparability of operations will be proved to be invariant by translations and we shall refer to the relations between the ordering and classification by fundamental translations uuder some additional condition.

Theorem 7. *If* $\lambda \gtrsim \mu$. *then* $\lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}$, $\lambda^{\underline{\mathfrak{x}}} \lesssim \mu^{\underline{\mathfrak{x}}}$ *for any* $\mathfrak{x} \in \mathfrak{G}$.

Proof. For any a, b, and $c \in E$.

$$(a \, \lambda^{\bar{\mathfrak{x}}} \, b) \, \mu^{\bar{\mathfrak{x}}} \, c = \left\{(a \, \lambda^{\bar{\mathfrak{x}}} \, b)^{\mathfrak{x}} \, \mu \, c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{(a^{\mathfrak{x}} \, \lambda \, b^{\mathfrak{x}}) \, \mu \, c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}}$$

$$= \left\{a^{\mathfrak{x}} \, \lambda \, (b^{\mathfrak{x}} \, \mu \, c^{\mathfrak{x}})\right\}^{\mathfrak{x}^{-1}} \qquad (\because \ \lambda \gtrsim \mu)$$

$$= \left\{a^{\mathfrak{x}} \, \lambda \, (b \, \mu^{\bar{\mathfrak{x}}} \, c)^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = a \, \lambda^{\bar{\mathfrak{x}}} \, (b \, \mu^{\bar{\mathfrak{x}}} \, c). \qquad \text{Hence} \quad \lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}.$$

$$(a \, \mu^{\underline{\mathfrak{x}}} \, b) \, \lambda^{\underline{\mathfrak{x}}} \, c = \left\{c^{\mathfrak{x}} \, \lambda \, (a \, \mu^{\underline{\mathfrak{x}}} \, b)^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{c^{\mathfrak{x}} \, \lambda \, (b^{\mathfrak{x}} \, \mu \, a^{\mathfrak{x}})\right\}^{\mathfrak{x}^{-1}}$$

$$= \left\{(c^{\mathfrak{x}} \, \lambda \, b^{\mathfrak{x}}) \, \mu \, a^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} \qquad (\because \ \lambda \gtrsim \mu)$$

---

6) $\{\mathfrak{e}\}$ is the set composed of only identity of $\mathfrak{G}$.

7) We can prove easily that $\lambda^{\mathfrak{p}} \in \mathfrak{U}$ if $\lambda \in \mathfrak{U}$.

$$=\left\{(b\,\lambda^{\mathfrak{x}}\,c)^{\mathfrak{x}}\,\mu\,a^{\mathfrak{x}}\right\}^{\mathfrak{x}-1}=a\,\mu^{\mathfrak{x}}\,(b\,\lambda^{\mathfrak{x}}\,c). \qquad \text{Hence} \qquad \lambda^{\mathfrak{x}} \lesssim \mu^{\mathfrak{x}}.$$

Definition 12. If there is a suitable $\mathfrak{x} \in \mathfrak{G}$ such that $\lambda = \mu^{\bar{\mathfrak{x}}}$, then λ and μ are said to be congruent, denoted $\lambda \equiv \mu$.

Since this binary relation $\equiv$ is obviously a equivalence relation, we can classify $\mathfrak{M}$ by it. This classification is called the classification of $\mathfrak{M}$ by $\bar{\mathfrak{G}}$, written $\mathfrak{M}/\bar{\mathfrak{G}}$, whose elements are classes $\mathcal{A}$, $\mathcal{B}, \cdots$ composed of operations. Here we call only $\mathfrak{U}/\bar{\mathfrak{G}}$ to account, into which a quasi-ordering is introduced similarly as that in $\mathfrak{U}$.

Definittion 13. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{U}/\bar{\mathfrak{G}}$. We denote $\mathcal{A} \gtrsim \mathcal{B}$ if for any $\lambda \in \mathcal{A}$ there exists one at least $\mu \in \mathcal{B}$ such that $\lambda \gtrsim \mu$.

It is evident that the binary relation $\mathcal{A} \gtrsim \mathcal{B}$ is a quasi-ordering in $\mathfrak{U}/\bar{\mathfrak{G}}$. By Definition 12 and Theorem 7 we readily obtain :

Theorem 8. *Definition 13, the following* (1), *and* (2) *are all equivalent.*

(1) *For any* $\mu \in \mathcal{B}$ *there exists one at least* $\lambda \in \mathcal{A}$ *such that* $\lambda \gtrsim \mu$.

(2) *There exist* $\lambda \ni \mathcal{A}$, $\mu \in \mathcal{B}$ *such that* $\lambda \gtrsim \mu$.

Now we are confined to the case that $\mathfrak{G}$ is finite.[8)]

Theorem 9. *If* $\lambda \equiv \mu$, *then either* $\lambda \sim \mu$ *or* $\lambda \not\gtrsim \mu$.

Proof. We suppose that λ and μ are comparable, say $\lambda \gtrsim \mu$. Since there exists $\mathfrak{x} \in \mathfrak{G}$ such that $\mu = \lambda^{\bar{\mathfrak{x}}}$ by Definition 12, it holds that $\lambda \gtrsim \lambda^{\bar{\mathfrak{x}}}$ ①; while, $\mathfrak{G}$ being finite,[9)] $\bar{\mathfrak{G}}$ is finite, whose order is n. Applying translations $\bar{\mathfrak{x}}, \bar{\mathfrak{x}}^2, \cdots, \bar{\mathfrak{x}}^{n-1}$ successively to both sides of ① by means of Theorem 7, we have $\lambda \gtrsim \lambda^{\bar{\mathfrak{x}}} \gtrsim \lambda^{\bar{\mathfrak{x}}^2} \gtrsim \cdots \gtrsim \lambda^{\bar{\mathfrak{x}}^n} = \lambda$, following that $\lambda \sim \mu$.

Theorem 10. *If there exist* $\lambda \in \mathcal{A}$ *and* $\mu \in \mathcal{B}$ *such that* $\lambda \gneqq \mu$,[10)] *then* $\xi \not\lesssim \eta$ *for any* $\xi \in \mathcal{A}$ *and* $\eta \in \mathcal{B}$.

Proof. Suppose that there exist $\xi \in \mathcal{A}$ and $\eta \ni \mathcal{B}$ such that $\xi \lesssim \eta$. Since $\mu = \eta^{\bar{\mathfrak{x}}}$ for a suitable $\mathfrak{x} \in \mathfrak{G}$, we have $\lambda \gtrsim \mu \gtrsim \xi^{\bar{\mathfrak{x}}}$. Theorem 9 shows that $\lambda \sim \xi^{\bar{\mathfrak{x}}}$, and so $\lambda \lesssim \xi^{\bar{\mathfrak{x}}}$, resulting in $\lambda \sim \mu$, which contradicts with the assumption that $\lambda \nsim \mu$.

Let $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \gtrsim \mathcal{B}$ as well as $\mathcal{A} \lesssim \mathcal{B}$.

Theorem 11. $\mathcal{A} \sim \mathcal{B}$ *if and only if there exist* $\xi \in \mathcal{A}$ *and* $\zeta \in \mathcal{B}$ *such that* $\xi \sim \zeta$.

Proof. Suppose that $\mathcal{A} \sim \mathcal{B}$. Then there exist $\lambda, \nu \in \mathcal{A}$ and $\mu, \eta \in \mathcal{B}$ such that $\lambda \gtrsim \mu$ and $\nu \lesssim \eta$. Having $\nu = \lambda^{\bar{\mathfrak{x}}}$ for some $\mathfrak{x}$ and $\lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}$ · it holds that $\mu^{\bar{\mathfrak{x}}} \lesssim \eta$,

8) The number of elements of $\mathfrak{G}$ is finiite.

9) By Theorem 5.

10) $\lambda \gneqq \mu$ symbols the fact that $\lambda \gtrsim \mu$ but $\lambda \nsim \mu$.

while $\mu^{\xi} \gtrsim \eta$ by Theorem 9 ; hence $\lambda^{\xi} \lesssim \mu^{\xi}$. Finally $\lambda^{\xi} \sim \mu^{\xi}$ where, of course, $\lambda^{\xi} \in \mathcal{A}$, $\mu^{\xi} \in \mathcal{B}$. The converse is needless to say.

§ 7. Some Necessary Conditions in Special Cases.

In this paragraph we shall arrange some necessary conditions which are fulfilled by a pair of comparable operations under the special assumptions. If E with the operation λ has a right (left) identity e or right (left) zero[11] n, then for the sake of simplicity we shall say that the operation λ has a right (left) identity e or a right (left) zero n respectively, or say that e or n is a right (left) identity or a right (left) zero of λ respectively.

Ideals [11] $I_\lambda^l(a)$, $I_\lambda^r(a)$ for $a \in E$ are defined as

$$I_\lambda^l(a) = [x\,\lambda\,a \mid x \in E], \quad I_\lambda^r(a) = [a\,\lambda\,x \mid x \in E].$$

Theorem 12.

(I) *If λ has a right identity and $\lambda \gtrsim \mu$, then $\mathfrak{R}_\mu \subset \mathfrak{R}_\lambda$ and $I_\mu^r(a) \subset I_\lambda^r(a)$ for eve every $a \in E$.*

(II) *If μ has a left identity and $\lambda \gtrsim \mu$, then $\mathfrak{L}_\lambda \subset \mathfrak{L}_\mu$ and $I_\lambda^l(a) \subset I_\mu^l(a)$ for every $a \in E$.*

Proof of (I). By the assumption, there is such an element e that $a\,\lambda\,e = a$ for every $a \in E$. Since $\lambda \gtrsim \mu$, we have $a\,\mu\,x = (a\,\lambda\,e)\,\mu\,x = a\,\lambda\,(e\,\mu\,x)$ for every $x \in E$. From this we get $\mathfrak{R}_\mu \subset \mathfrak{R}_\lambda$ and $I_\mu^r(a) \subset I_\lambda^r(a)$.

Theorem 13.

(I) *The element e is a right identity of λ as well as a left identity of μ. Then either $\lambda = \mu$ or $\lambda \not\gtrless \mu$.*

(II) *The element e is a left identity of λ as well as a right identity of μ. Then either $\lambda = \mu$ or $\lambda \not\gtrless \mu$.*

Proof. of (I). Suppose $\lambda \gtrsim \mu$, then $x\,\mu\,y = (x\,\lambda\,e)\,\mu\,y = x\,\lambda\,(e\,\mu\,y) = x\,\lambda\,y$ for every x and $y \in E$. Hence $\lambda = \mu$.

Theorem 14. *If the element e is the two-sided identity of both λ and μ, then either $\lambda = \mu$ or $\lambda \not\gtrless \mu$.*

Proof. Suppose $\lambda \gtrsim \mu$ or $\lambda \lesssim \mu$, then we have $x\,\lambda\,y = x\,\mu\,y$ for every x and $y \in E$; hence $\lambda = \mu$.

Theorem 15.

(I) *If $\lambda \gtrsim \mu$ then a right zero of μ implies a right zero of λ.*

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11) By a right zero n of E is meant such an element n that $x\,\lambda\,n = n$ for all $x \in E$.

(II) *If* $\lambda \gtrsim \mu$ *then a left zero of* λ *implies a left zero of* μ.

Proof. of (I) Let n be a right zero of μ. $x \lambda n = x \lambda (y \mu n) = (x \lambda y) \mu n = n$.

Theorem 16.

(I) *If* $\lambda \gtrsim \mu$ *and* n *is a right zero of* λ, *then* $n \mu y$ *for* $y \in E$ *is a right zero of* λ.

(II) *If* $\lambda \gtrsim \mu$ *and* n *is a left zero of* μ, *then* $x \lambda n$ *for* $x \in E$ *is a left zero of* μ.

Proof of (I) For every $x \in E$, $x \lambda (n \mu y) = (x \lambda n) \mu y = n \mu y$.

Theorem 17.

(I) *If* $\lambda \gtrsim \mu$ *and* n *is the only right zero of* λ, *then* n *is a left zero of* μ.

(II) *If* $\lambda \gtrsim \mu$ *and* n *is the only left zero of* μ, *then* n *is a right zero of* λ.

Proof of (I) By Theorem 16 (I), $n \mu y$ for $y \in E$ is a right zero of λ. From the uniqueness of right zero follows $n \mu y = n$ for every $y \in E$.

Notes.

[1] The study of semigroups semigroups has been achieved by many mathematcians, Arnold, Lorenzen, Clifford, Suschkewitch, etc., but I have not yet read their works. With respect to the representation of semigroups, see

E. Hille: Functional analysis and semi-groups, 1946, p. 147.

[2] Let P, Q, R be transformations of a set M. By the definition of product,

$$\{(PQ)R\}x = R\{(PQ)x\} = R\{Q(Px)\}, \ \{P(QR)\}x = (QR)(Px) = R\{Q(Px)\},$$

and so $\{(PQ)R\}x = \{P(QR)\}x$ for every $x \in M$. Hence $(PQ)R = P(QR)$.

[3] If $E(\lambda)$ is a semigroup,

$$\{R_\lambda(a) R_\lambda(b)\}x = R_\lambda(b)\{R_\lambda(a)x\} = R_\lambda(b)(x \lambda a) = (x \lambda a) \lambda b$$

$$= x \lambda (a \lambda b) = R_\lambda(a \lambda b)x; \text{ therefore } R_\lambda(a) R_\lambda(b) = R_\lambda(a \lambda b).$$

Similarly $L_\lambda(a) L_\lambda(b) = L_\lambda(b \lambda a)$; hence if $E(\lambda)$ is a semigroup, then $\mathfrak{R}_\lambda$ aud $\mathfrak{L}_\lambda$ are algebraic subsystems of $\mathfrak{T}_E$, and conseqruntly semigroups. However this converse is not true unless, the correspondence $a \leftrightarrow R_\lambda(a)$ is one-to-one.

[4] We suppose that E is a semigroup. Evidently $(x \bar{\lambda} y) \bar{\lambda} z = x \bar{\lambda} (y \bar{\lambda} z)$ for $x, y, z \in E$; by the definition of $\bar{\lambda}$, $(p \bar{\lambda} x) \bar{\lambda} y = p \bar{\lambda} (x \bar{\lambda} y)$, $(x \bar{\lambda} p) \bar{\lambda} y = x \bar{\lambda} (p \bar{\lambda} y)$, and $(x \bar{\lambda} y) \bar{\lambda} p = x \bar{\lambda} (y \bar{\lambda} p)$ for $x, y \in \bar{E}$. Thus $\bar{E}$ is a semigroup. The converse is proved by Lemma 1 and 2.

〔5〕〔6〕 Birkoff : Latfice theory, 1948, *p.* 4.

〔7〕 There are cases that the transitive law holds, even if no universality is assumed. For example,

ξ

	a	b
a	a	b
b	a	b

η

	a	b
a	a	a
b	a	a

ζ

	a	b
a	a	a
b	b	b

where surely $\xi \gtrsim \eta$, $\eta \gtrsim \zeta$ and $\xi \gtrsim \zeta$.

〔8〕 We can prove them not by Theorem 1, but directly by the product tables. In greater detail, Takayuki Tamura : On the condition for semigroup (Japanese), Shikoku Sugaku Danwa, No. 2, 1951.

〔9〕 Furtheremore we have $\lambda \not\equiv \nu$.

〔10〕 In reality it holds that $\mathfrak{A}(\lambda^{\bar{\mathfrak{x}}})=\mathfrak{A}(\lambda)$ for any $\bar{\mathfrak{x}} \in \bar{\mathfrak{S}}$.

〔11〕 Takayuki Tamura, Characterization of groupoids and semilattices by idealds in a semigroup, Journal of Science of the Gakugei Faculty Tokushima University, Vol 1, 1950, p. 37.

August 1951,

Gakugei Faculty,
Tokushima University.

Addendum to the paper "On a relation between local convexity and entire convexity." in this Journal, vol. 1.

In p. 25. vol. 1. I defined "convex point x of M", which is explained additionally as following.

If there exists $\delta > 0$ such that $U(x;\varepsilon) \cap M$, as far as non-null, is con ex for any positive $\varepsilon \leqq \delta$, *the point of the space* Ω *is called a convex point regarding* M, *or* M *is said to be (locally) convex at* x.

Furthermore I correct the errors in the same paper as below.

	error	correct
line 4, page 29,	for any $\varepsilon > 0$	for a sufficiently small $\varepsilon > 0$
last line page 29,	for some $\gamma < 0$	for some ξ_0, $\gamma > 0$

Notes on General Analysis (I)

By

Isae SHIMODA

(Received Sept. 30, 1951)

In these notes we shall first give another proofs of the radius of analyticity of the power series[*)] which term by term differentiated and the Taylor expansion of the power series in the sphere of analyticity,[1)] and then investigate in detail the state of the singular point of the power series on the boundary of the sphere of analyticity.[2)] In the end of these papers, we shall extend the theorem of Osgood of two complex variables to the case of functions whose domains lie in product spaces of two complex Banach spaces using the classical methods.

§ 1. Radius of analyticity of the power series,

Let E_1, E_2 and E_3 be complex-Banach-spaces and an E_2 valued function $h_n(x)$ defined on E_1 be a homogeneous polynomial of degree n. Then the radius of analyticity τ of the power series $\sum_{n=0}^{\infty} h_n(x)$ is given by

$$\frac{1}{\tau} = \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|}\ ^{2)}.$$

We shall use following lemma for our purpose.

Lemma. *Suppose that x and y are arbitrary points respectively on $\|x\|<\tau$ and on $\|y\|=1$. Let ρ be an arbitrary positive number such that $\rho<\tau-\|x\|$. Then there exists a positive number σ which is less than 1 and satisfies the following inequalities*

$$\| h_n(x+\alpha y) \| < \sigma^n$$

for $|\alpha| \leq \rho$ and $n \geqslant n_0(\rho, \sigma)$.[2)]

Put $h_n(x+\alpha y) = \sum_{i=0}^{n} h_{n-i,i}(x, y)\alpha^i$. Then $h_{n-i\cdot i}(x, y)$ is a homogeneous polynomial of

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*) This is called "The radius of absolute convergence of the power series" by E. Hille ; Functional analysis and semigroups, 1948.

1) See, A. E. Taylor, (1) Analytic functions in general analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. 11 Vol. vl (1937). (2) Additions to the theory of polynomials in normed linear spaces (Tohoku M. J. 44, 1938). (3) On the properties of analytic functions in abstract spaces, Math. Ann. 115, 1938.

2) I. Shimoda : (1) On power series In abstract spaces. Mathematica Japonicae, Vol. 1, No. 2. The principal part of the proof of Theorem 2 is "Lemma" in this paper. (2) On the behaviour of power series on the boundary of the sphere of analyticity in abstract spaces, Proceeding of Japan A. Vol. 27 (1951), No. 2. or, Journal of Science of Gakugei Faculty, Tokusima University, Vol. 1, 1950.

degree n-i with respect to x and a homogeneous polynomial of degree i with respect to y. $h_{n-1,1}(x, y)$ is the differential of $h_n(x)$ with increment y.

Theorem 1. *The radius of analyticity of* $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ *with respect to* x *and independent of* y *is* τ.

Proof. The radius of analyticity τ' of $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ independent of y is clearly

$$\frac{1}{\tau'} = \sup_{y} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n-1]{\|h_{n-1,1}(x, y)\|}.$$

Now put $y = \|y\| \cdot y'$, then

$$\frac{1}{\tau'} = \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n-1]{\|h_{n-1,1}(x, y')\| \cdot \|y\|}$$

$$= \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, y')\|}.$$

When $x = y'$, $h_{n-1,1}(x, x) = nh_n(x)$.[3] Therefore

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} = \overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{n}} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, x)\|}$$

$$\leqq \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, y')\|}$$

and we have $\frac{1}{\tau} \leqq \frac{1}{\tau'}$. That is, $\tau \geqq \tau'$.

Let x and y be arbitrary points respectively in $\|x\| < \tau$ and E_1. Since there exists a positive number ρ' such that $0 < \rho' \|y\| < \tau - \|x\|$, we have

$$\| h_n(x + \alpha y) \| < \sigma^n,$$

for $|\alpha| \leqq \rho'$ and $n \geqq n_0$, by lemma, where $0 < \sigma < 1$. Thus we have, $\| h_{n-1,1}(x, y) \| < \frac{1}{\rho'} \sigma^n$, for $n \geqq n_0$. This shows that $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ is absolutely convergent in $\|x\| < \tau$ for an arbitrary fixed y and we see that $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ is analytic in $\|x\| < \tau$. That is, $\tau \leqq \tau'$. Then we have $\tau = \tau'$.

Corollary. Let $h_{n-i,i}(x, y_1, y_2, \ldots, y_i)$ be the i-th derivative of $h_n(x)$ with increments $y_1, y_2, \ldots y_i$. Then the radius of analyticity of $\sum_{n=i}^{\infty} h_{n-i,i}(x, y_1, y_2, \ldots, y_i)$ with respect to x and independent of $y_1, y_2, \ldots, y_i$ is τ.

Theorem 2. *Let* x *be an arbitrary point in* $\|x\| < \tau$. *Then the radius of analyticity of the Taylor expansion of* $\sum_{n=0}^{\infty} h_n(x)$ *at* x *is greater than or equal to* $\tau - \|x\|$.

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3) A. E. Taylor (2), Theorems 2. 3, 2. 5 and 2. 7.

Proof. Let y be an arbitrary point on $\|y\|=1$, and ρ be an arbitrary positive number such that $\rho < \tau - \|x\|$.

Appealing to Lemma, we have

$$\| h_n(x+\alpha y) \| < \sigma^n,$$

for $|\alpha| \leqq \rho$ and $n \geqslant n_0$, where $0 < \sigma < 1$. Then we have,

$$\| h_{n-i,i}(x, y) \| \leqq \frac{1}{\rho^i} \sigma^n \cdots\cdots\cdots\cdots (1)$$

for $\|y\|=1$ and $n \geqslant n_0$. Now, put $U_m(y) = \sum_{n=i}^{m} h_{n-i,i}(x, y)$ with $m = i, i+1, \ldots\ldots$ and for an arbitrary y in complex Banach spaces. $U_m(y)$ is a homogeneous polynomial of degree i satisfying the following inequalities

$$\| U_p(y) - U_q(y) \| = \| \sum_{n=q+1}^{p} h_{n-i,i}(x, y) \|$$

$$= \|y\|^i \cdot \| \sum_{n=q+1}^{p} h_{n-i,i}(x, y') \|, \text{ where } y' = \frac{y}{\|y\|},$$

$$\leqq \|y\|^i \cdot \sum_{n=q+1}^{p} \frac{1}{\rho^i} \sigma^n, \text{ from } (1),$$

$$\leqq \left(\frac{\|y\|}{\rho}\right)^i \frac{1}{1-\sigma} \sigma^{q+1},$$

for $p > q \geqslant n_0$. This shows that the sequence $\{U_m(y)\}$ is convergent on whole spaces, and we see that $\lim_{m\to\infty} U_m(y) = \sum_{n=i}^{\infty} h_{n-i,i}(x, y)$ is a homogeneous polynomial of degree i with respect to y.[4] Put $h'_i(y) = \sum_{n=i}^{\infty} h_{n-i,i}(x, y)$ and let τ' be the radius of analyticity of the power series $\sum_{i=0}^{\infty} h'_i(y)$, then we have

$$\frac{1}{\tau'} = \sup_{\|y\|=1} \overline{\lim_{i\to\infty}} \sqrt[i]{\| h'_i(y) \|}$$

$$\leqq \sup_{\|y\|=1} \overline{\lim_{i\to\infty}} \sqrt[i]{\frac{1}{\rho^i} \frac{\sigma^i}{1-\sigma}}, \text{ from } (1),$$

$$\leqq \frac{\sigma}{\rho} < \frac{1}{\rho}.$$

Thus we have $\tau' \geqslant \rho$. Since $\tau - \|x\| - \rho$ can be taken as small as we like, we have $\tau' \geqslant \tau - \|x\|$. This completes the proof.

§ 2. Singular point of power series**)

The radius of analyticity τ of the power series $\sum_{n=0}^{\infty} h_n(x)$ is given by following equation[5]

$$\frac{1}{\tau} = \sup_{G \in K} \overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x \in G} \| h_n(x) \|},$$

4) A. E. Taylor (2), Theorem 3. 7.

5) See : (2) of 2).

where G is an arbitrary compact set extracted from the set $\|x\|=1$ and K is composed of all such compact sets. The sphere $\|x\| < \tau$ is called the sphere of analyticity of $\sum_{n=0}^{\infty} h_n(x)$.

Theorem 3. *Suppose that a compact set G exists on the boundary of the sphere of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$, which satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \| h_n(x) \|} = 1 . \quad \cdots\cdots(2)$$

Then we can find a sequence $\{x_n\}$, which converges to x_0 and satisfies the equation $\overline{\lim_{n_i\to\infty}} \sqrt[n_i]{\|h_{n_i}(x_i)\|} = 1$, in G and at least a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on the set M composed of $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

Proof. From the assumption (2), we have

$$\sqrt[n_i]{\sup_{x\in G} \| h_{n_i}(x) \|} > \frac{1}{1+\varepsilon_i}, \quad \cdots\cdots(3)$$

for a sequence of positive number ε_i, which tends to zero, where n_i depends on ε_i for $i=1, 2, \ldots\ldots$

Since G is compact and $h_{n_i}(x)$ is continuous on C, there exists x_i in G which satisfies

$$\| h_{n_i}(x_i) \| = \sup_{x\in G} \| h_{n_i}(x) \| .$$

Since $\{x_i\}$ is a subset of G, we can select a subsequence of $\{x_i\}$ which converges in G. In order not to change notation, we shall suppose simply that the sequence $\{x_i\}$ itself converges to x_0, which is the element of G. Then, from the construction of $\{n_i\}$, we have

$$\lim_{n_i\to\infty} \sqrt[n_i]{\| h_{n_i}(x_i)\|} = 1, \quad \text{and} \quad x_i \longrightarrow x_0 .$$

Put $x_i(1+\varepsilon_i)=y_i$, then y_i converges to x_0. From (3), we have

$$\| h_{n_i}(y_i) \| > 1 \quad \cdots\cdots(4)$$

If $\sum_{n=0}^{\infty} h_n(x)$ has not a singular point on M, which composed of $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$, $\sum_{n=0}^{\infty} h_n(x)$ is analytic on M. Therefore, for an arbitrary positive number ε and θ $(0 \leqq \theta \leqq 2\pi)$, there exists N_θ, such that

$$\| \sum_{n=0}^{\infty} h_n(x) \| \leqq N_\theta ,$$

for $\| x-x_0 e^{i\theta} \| < \varepsilon$. By the covering theorem of Heine-Borel for a compact set, we can find finite points $x_0 e^{i\theta_1}$, $x_0 e^{i\theta_2}$, ..., $x_0 e^{i\theta_m}$, such that $\| \sum_{n=0}^{\infty} h_n(x) \| \leqq N$, for $\|x-x_0 e^{i\theta_j}\| \leqq \varepsilon$, where $j=1, 2, 3, \ldots, m$ and $N=\max(N_{\theta_1}, N_{\theta_2}, \ldots, N_{\theta_m})$. Now we choose two positive numbers ρ and δ, such that $\|\alpha x - x_0 e^{i\theta_j}\| < \varepsilon$, when $\| x-x_0 \| < \rho (< \varepsilon)$, $|\alpha| = 1+\delta$

and suitable θ_j is chosen from $\theta_1, \theta, \ldots, \theta_m$ for α.

Then we have

$$\| h_n(x) \| = \left\| \frac{1}{2\pi i} \int_{|\alpha|=1+\delta} \frac{\sum_{n=0}^{\infty} h_n(x)}{\alpha^{n+1}} d\alpha \right\| \leqq \frac{N}{(1+\delta)^n} \cdots\cdots\cdots\cdots(5)$$

for $\|X-X_0\| < \rho$ and $n=1, 2, \ldots\ldots$

Since y_i converges to X_0, (5) contradicts to (4). This shows that $\sum_{n=0}^{\infty} h_n(x)$ has at least a singular point on M. Here x is not necessarily a singular point of $\sum_{n=0}^{\infty} h_n(x)$, as a following example shows. Put $h_n(X) = x^{n-1} y$ in the complex-2-dimensional spaces, then $h_n(X)$ is a homogeneous polynomial of degree n, where $X=(x, y)$. Then the radius of analyticity of $\sum_{n=1}^{\infty} h_n(x)$ is 1. Let G be a compact set on $\|X\|=1$ composed of $X_0=(e^{i\theta}, 0)$ and $X_m=\left(\sqrt{1-\frac{1}{m}}\, e^{i\theta}, \sqrt{\frac{1}{m}}\, e^{i\theta}\right)$ with $m=1, 2, 3, \ldots\ldots$

$$\sup_{X \in G} \| h_n(X) \| = \sup_m \left| \left(1-\frac{1}{m}\right)^{\frac{n-1}{2}} e^{i(n-1)\theta} \left(\frac{1}{m}\right)^{\frac{1}{2}} e^{i\theta} \right|$$

$$= \sup_m \left(1-\frac{1}{m}\right)^{\frac{n-1}{2}} \left(\frac{1}{m}\right)^{\frac{1}{2}}$$

$$= \left(1-\frac{1}{n}\right)^{\frac{n-1}{2}} \left(\frac{1}{n}\right)^{\frac{1}{2}}$$

because $(1-t)^{n-1} t$ takes its maximum at $t=\frac{1}{n}$ in the interval $0 \leqq t \leqq 1$. Since $\left(1-\frac{1}{n}\right)^{\frac{n-1}{2}} \left(\frac{1}{n}\right)^{\frac{1}{2}} = \| h_n(X_n) \|$, we have

$$\sup_{X \in G} \| h_n(X) \| = \| h_n(X_n) \| .$$

On the other hand, $\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X_n) \|} = \overline{\lim_{n\to\infty}} \left(1-\frac{1}{n}\right)^{\frac{n-1}{2n}} \left(\frac{1}{n}\right)^{\frac{1}{2n}} = 1$, and moreover X_m converges to X_0. Nevertheless, it is not X_0 but $X_0 e^{-i\theta}$, that is a singular point of $\sum_{n=1}^{\infty} h_n(X)$.

Corollary. *Suppose that a compact set G exists on $\|x\|=1$ such that*

$$\frac{1}{\tau} = \overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{X \in G} \| h_n(X) \|} ,$$

then we can find a sequence $\{x_n\}$, which converges to x_0 in G and satisfies $\overline{\lim_{n\to\infty}} \sqrt[n_i]{\| h_{n_i}(X_i) \|} = \frac{1}{\tau}$, and at least a singular point on the set composed of $x_0 \tau e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

Theorem 4. *If a point x, which lies on the boundary of the sphere of analiticity of $\sum_{n=0}^{\infty} h_n(X)$, satisfies the following equality $\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X) \|} = 1$,* *then there exists at least a singular point on the circle* $x e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

Proof. Since x is a compact set, $\sup \| h_k(X) \| = \| h_k(X) \|$ and $\{x\}$ converges to x. Therefore, Theorem 3 is applicable and we see that Theorem 4 is true.

Corollary, *If a point* x, *which lies on* $\| x \| = 1$, *satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X) \|} = \frac{1}{\tau},$$

then there exists at least a singular point on the circle $x \tau e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

As well as the case of Theorem 3, $x \tau$ is not necessarily a singular point, as we can easily find an opposite example in the power series of complex numbers.

§ 3. Analytic functions of two variables

Lemma. *Let* $\{f_n(x)\}$ *be a sequence of functions on* E_1 *to* E_2, *each being analytic in a domain D, and convergent to* $f(X)$ *in D. If on each compact set G extracted from D the members of the sequence possess a commom bound M,* $f(x)$ *is analytic in D.*

Proof. Let x_0 be any point in D, then there exists a pair of positive numbers ρ, M, for which $\| f_n(x) \| \leqq M$ $(n=1, 2, \ldots)$, when $\|x-x_0\| \leqq \rho$ in D. If not so, there exists a subsequence $\{f_m(x)\}$ of $\{f_n(x)\}$ and a sequence $\{x_m\}$, which tends to x_0, such that $\| f_m(x_m) \| \geqq m$. On the other hand, since $\{x_m\}$ is a compact set, $f_n(x)$ must be bounded on $\{x_m\}$ in contradiction to $\| f_m(x_m) \| \geqq m$. Then $f(x)$ is analytic in $\| x-x_0 \| < \rho$ by the thorem of A. E. Taylor.[6] Therefore $f(x)$ is analytic in D.

Theorem 5. *A function* $f(x, y)$ *defined in a domain D of* $E_1 \times E_2$ *with values of* E_3 *is analytic in D if the following conditions are satisfied,* 1) $f(x, y)$ *is analytic with respect to* x, y *separately in D,* 2) *let G be any compact set extracted from D, then there exists a positive number* M_G *such that* $\| f(x, y) \| \leqq M_G$ *on G.*

Proof. Let (x_0, y_0) be any point of D. We can choose two positive numbers R, S such that a domain $\| x-x_0 \| < R$, $\| y-y_0 \| < S$ is contained in D. Then it suffices to show

6) A. E. Taylor (3), loc. cit. page 469. Theorem 15. Let $\{f_n(x)\}$ be a sequence of functions on E_1 to E_2, each analytic in a domain D of E_1, and convergent to a limit $f(x)$ in D. If in each region interior to D the members of the sequence possess a common bound, $f(x)$ is analytic in D.

that $f(x, y)$ is analytic in a domain $\| x-x_0 \| < R$, $\| y-y_0 \| < S$. Without losing generality, we may assume that $(x_0, y_0)=(0, 0)$. If x is an arbitrary fixed point of $\|x\| < R$, $f(x, y)$ is an analytic function of y in $\|y\| < S$. Therefore we have

$$f(x, y) = \sum_{n=0}^{\infty} U_n(x, y),$$

where $U_n(x, y)$ is a homogeneous polynomial of degree n with respect to y. Obviously $U_0(x, y)=f(x, 0)$, which is analytic with respect to x. If y is an arbitrary fixed point of $\|y\| < S$, there exists a positive number ρ such that $\rho \|y\| < S$. Then we have

$$U_1(x, y)=\frac{1}{2\pi i}\int \frac{f(x, \alpha y)}{\alpha^2} d\alpha,$$

the integral being taken in the positive sense on the circle $|\alpha|=\rho$. Now we define $S_m(x)=\frac{1}{2\pi i}\left\{\sum_{i=1}^{m}\frac{f(x, \eta_i y)}{\eta_i^2}(\xi_{i+1}-\xi_i)\right\}$ for $m=1, 2, \ldots,$ where $\xi_1, \xi_2, \ldots, \xi_m, \xi_{m+1}(=\xi_1)$ lie on the circle $|\alpha|=\rho$, and each η_i lies on the arc $\widehat{\xi_i \xi_{i+1}}$, and $\max_{1 \leq i \leq m} |\xi_{i+1}-\xi_i|$ tends to zero when m tends to infinity. Then

(1), $S_m(x)$ is analytic in $\|x\| < R$,

(2). if x is an arbitrary fixed point of $\|x\| < R$, $\lim_{m\to\infty} S_m(x)=U_1(x, y)$,

(3). let G_0 be any compact set extracted from the sphere $\|x\| < R$, and T be a set of αy, where $|\alpha|=\rho$, $G=(G_0, T)$ is a compact set in D. By the hypothesis 2) there exists a positive number M_G such that $\| f(x, \alpha y) \| \leq M_G$, for $(x, \alpha y)$ on G. Therefore

$$\| S_m(x) \| \leq \frac{M_G}{\rho}.$$

Thus the lemmer is applicable, and we see that $U_1(x, y)$ is analytic with respect to x. On the other hand, $U_1(x, y)$ is linear with respect to y, and we see by the theorem of Kerner[7] that $U_1(x, y)$ is continuous in $(\|x\| < R, E_1')$. Generally $U_n(x, y)=\frac{1}{n!}\left[\delta^n f(x; y_1, y_2, \ldots, y_n)\right]$, where

$$\delta^n f(x; y_1, y_2, \ldots, y_n)=\frac{1}{(2\pi i)^n}\int\frac{d\alpha_1}{\alpha_1^2}\int\frac{d\alpha_2}{\alpha_2^2}\cdots\cdots\int\frac{f(x, \alpha_1 y_1+\cdots\cdots+\alpha_n y_n)}{\alpha_n^2} d\alpha_n,$$

each integral being taken in the positive sense on the circle $|\alpha_i|=\rho'$ for $i=1, 2, \ldots, n$, where ρ' must satisfy $\|\alpha_1 y_1+\cdots\cdots+\alpha_n y_n\| < S$ when $|\alpha_i| \leq \rho'$ $(i=1, 2, \ldots, n)$. Repeating the process described in the proof of the continuity of $U_1(x, y)$, we see that $\delta^n f(x; y_1, y_2, \ldots, y_n)$ is continuous with respect to $(x, y_1, y_2, \ldots, y_n)$. Now let G be any compact set extracted from $(\|x\| < R, \|y\| < S)$, and G_0, G_1 be the projections of G into $\|x\| < R$ and $\|y\| < S$ respectively. Since G_0, G_1 are clearly the compact sets in $\|x\| < R$, $\|y\| < S$ respectively, it follows that $\max_{G \ni y} \|y\|=s < S$. Let C be a circle $|\alpha|=\rho$ (where $1 < \rho < \frac{S}{s}$), $C\times G_1$ is a

7) M. Kerner, Zur Theorie der impliziten funktional Operation, Studia Math. T. III. (1931)

compact set in $\|y\| < S$ and $G' = (G_0, C \times G_1)$ is a compact set in $(\|x\| < R, \|y\| < S)$. Then there exists a positive number $M_{G'}$ such that $\| f(x, y) \| < M_{G'}$ when $(x, y) \in G'$. Obviously G is contained in G', and this shows

$$\| U_n(x, y) \| = \| \frac{1}{2\pi i} \int_c \frac{f(x, \alpha y)}{\alpha^{n+1}} d\alpha \| \leq \frac{M_{G'}}{\rho^n}$$

for $n = 1, 2, 3, \ldots$, when $(x, y) \in G$. Thus the function $f(x, y) = \sum_{n=0}^{\infty} U_n(x, y)$ converges uniformly on G, and so $f(x, y)$ is continuous in $(\|x\| < R, \|y\| < S)$. This completes the proof.

Corollary. *If E_2-valued function $f(x, y)$ is analytic with respect to x, y separately and bounded in the domain D of $E_1 \times E_2$, $f(x, y)$ is analytic in D.*

Remark. By using Theorem 5 and the theorem of B-continuity of Zorn,[8] *the generalized Hartogs's theorem* can be proved as in the classical methods. Let $f(x, y)$ is analytic with respect to each variables separately, then there exists an open set V, in which $f(x, y)$ is bounded, in an arbitrary neighbourhood U of any point (x, y) in the domain. Appealing to Theorem 5, $f(x, y)$ is analytic in V. Therefore $f(x, y)$ is B-continuous and then $f(x, y)$ is analytic with respect to (x, y) by the Theorem of Zorn, because $f(x, y)$ is G-differentiable.

**) A power series $\sum_{n=0}^{\infty} h_n(x)$ is called analytic at a point x when there exists at least a neighbourhood $V(x)$ of x, on which $\sum_{n=0}^{\infty} h_n(x)$ is continuous strongly and G-differentiable. A point x is called a singular point of $\sum_{n=0}^{\infty} h_n(x)$, when $\sum_{n=0}^{\infty} h_n(x)$ is not analytic at a point x.

8) Max A. Zorn : Characterization of Analytic Functions in Banach Spaces, Annals of Math. (2) 46 (1945). In the paper, the generalized Hartogs's theorem was proved very elegantly by Zorn.

On the ω^2 Distribution

By

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As a criterion for testing the goodness of fit of frequency function etc., hitherto K. Pearson's χ^2 test was extensively applied. However, in the quantity

$$\chi^2 = \Sigma w_\nu (m_\nu - Np_\nu)^2,$$

the weight being assumed to be $w_\nu = 1/Np_\nu$, those data with less probabilities have unreasonably more effect. On the contrary, the ω^2 test, originally due to R.v. Mises and H. Cramér and improved by N. Smirnoff[1)] is quite free from this defect. Nevertheless its table seems not yet to have been found. For this requirement, Y. Ueda constructed a table of ω^2_∞ distribution, as his graduation thesis at Waseda University, under the direction of the author. But it appears to be rather desirable to prepare those of $\omega_n{}^2$ distributions, n being the number of classes in any statistics, say 5～25. In the present note, however, it is only theoretically developped how to obtain the ω_n distribution parallel to Smirnoff's ω^2_∞ distribution, while the actual numerical computations as to ω_n are about going to be executed by few students in our institute, and some results might be expected.

§ 1. *Definitions.* Let some random variable x be subject to the probability density function $f(x)$, and the cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

Let $x_1 \leqq x_2 \leqq \cdots\cdots \leqq x_N$ be the observed values of x in N experimental trials, and

$$S(x) = NS_N(x)$$

denote the number of individuals which do not exceed a given value x, so that $S_N(x) = S(x)/N$ is an empirical accumlated probability, and increases in a stairway. Accordingly

$$\delta(x) = NS_N(x) - NF(x)$$

gives the deviation at $x = x$ of the experimental value from the theoretical. Now we define

$$\omega^2 = \frac{1}{N}\int_{-\infty}^{\infty} w(x)\ \delta(x)^2\ dx = N\int_{-\infty}^{\infty} w(x)\left[S_N(x) - F(x)\right]^2 dx, \qquad (1)$$

where $w(x)$ is the weight. Thus ω^2 affords the degree of deviation, and the smaller it is, the more precisely the assumed function represents the actual feature (R. v. Mises). Further N.

1) N. Smirnoff : Sur la distribution de ω^2, Comptes Rendus, 202 (1936), p. 449.

Smirnoff adopted the following form :

$$\omega^2 = N\int_{-\infty}^{\infty} w(F(x))\left[S_N(x)-F(x)\right]^2 F'(x)\,dx, \tag{2}$$

and in particular, taking the weight $w(F(x))=1$,

$$\omega^2 = N\int_{-\infty}^{\infty}\left[S_N(x)-F(x)\right]^2 f(x)\,dx, \qquad (f(x)=F'(x)). \tag{3}$$

The last form can be obtained by putting $w(x)=f(x)$ in (1), and thus the weight being directly propotional to the frequency $f(x)$, it is more legitimate than χ^2, in which the weight is inversely proportional to $f(x)$. Furthermore, while v. Mises' original ω^2 is subjected to the selection of $F(x)$, Smirnoff's ω^2, as Stieltjes integral, is quite independent of it, and therefore could be evenly applied for any distribution.

§ **2.** *The distribution function of* ω^2_n, $\Phi(\omega^2_n)$. Smirnoff's ω^2 being independent of the choice of F, we may after him assume the case of the uniform distribution in $\langle 0, 1\rangle$, so that

$$\begin{aligned} f(x)&=0, & F(x)&=0, &&\text{for } x<0,\\ &=1, & &=x, &&\text{for } 0\leqq x\leqq 1,\\ &=0, & &=1, &&\text{for } x>1. \end{aligned}$$

Evidently the theoretical probability that an observed point falls in the partial interval

$$\frac{k-1}{n}\leqq x\leqq\frac{k}{n} \qquad (k=1,\ 2,\cdots\cdots,\ n) \tag{4}$$

is $p_k=1/n$, and the cumulative probability that x does not exceed l/n is l/n. Hence, if m_k be the empirical number of points having fallen in the interval (4) in N trials, then Pearson's weighted deviation[1] becomes

$$t_k=\left(m_k-Np_k\right)\Big/\sqrt{Np_k}=\left(m_k-\frac{N}{n}\right)\Big/\sqrt{\frac{N}{n}}, \tag{5}$$

where $k=1,\ 2,\cdots\cdots,\ n-1$, and $t_n=-\sum_{\nu=1}^{n-1}t_\nu$.

If N be sufficiently great, and n tolerably large, say $n=20$, the integral (2) may be approximated by summation concerning n intervals (4) as follows :

$$\begin{aligned}\omega^2_{N,n}&=\frac{N}{n}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\left[\frac{m_1+\cdots\cdots+m_l}{N}-\frac{l}{n}\right]^2=\frac{1}{nN}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\left[\sum_{k=1}^{l}\left(m_k-\frac{N}{n}\right)\right]^2\\ &=\frac{1}{nN}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\frac{N}{n}\left(\sum_{k=1}^{l}t_k\right)^2=\frac{1}{n^2}\sum_{j,k}^{n-1}t_j\,t_k\sum_{l=j}^{n-1}w\left(\frac{l}{n}\right),\end{aligned} \tag{6}$$

1) As we have remarked before, Pearson's weight is inadequate, yet we may utilize it merely for the sake of convenient transformation.

where $g = \text{Max}\ (j, k)$. That is

$$\omega^2_{N,n} = \sum_{j,k=1}^{n-1} a_{jk}\ t_j\ t_k, \quad \text{where} \quad a_{jk} = a_{kj} = \frac{1}{n^2} \sum_{l=g}^{n-1} w\left(\frac{l}{n}\right).$$

Specially for $w = 1$, we have $a_{jk} = a_{kj} = \dfrac{n-g}{n^2}$, and therefore

$$\omega^2_{N,n} = \sum_{j,k=1}^{n-1} \frac{n-g}{n^2} t_j\ t_k = \sum a_{jk}\ t_j\ t_k = A\ (t,\ t). \tag{7}$$

Thus our ω^2 is reduced to a positive definite Hermite form.

Now we require to find the c. d. f. $\Phi_{N,n}\ (\omega^2)$, or its p. d. f. $\varphi\ (\omega^2)$. For this purpose, we shall begin to determine its characteristic

$$\psi_{N,n}\ (\xi) = \sum_{m_1+\cdots+m_n=N} p\ (m_1, \cdots\cdots, m_n)\ \exp\left\{i\ \xi\ A\ (t, t)\right\}, \quad i = \sqrt{-1}, \tag{8}$$

where p denotes the probability that the respective number of points falling in the k-th subinterval becomes m_k $(k = 1, \cdots\cdots, n)$.

When $N \longrightarrow \infty$, as is well known in χ^2-distribution, the expression (8) shall tend to the limit :

$$\lim_{N\to\infty} \psi_{N,n}\ (\xi) \equiv \psi_n(\xi) = \frac{\sqrt{n}}{\sqrt{2\pi}^{\,n-1}} \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\chi^2 + i\ \xi\ \ A\ (t, t)\right\} dt_1 \cdots dt_{n-1}, \tag{9}$$

where $\chi^2 = \sum_{k=1}^{n} t_k{}^2 = \sum_{k=1}^{n-1} t_k{}^2 + \left(-\sum_{k=1}^{n-1} t_k\right)^2 = 2\sum_{k=1}^{n-1} t_k{}^2 + \sum_{j \neq k}^{n-1} t_j\ t_k.$

Hence the exponent in (9) becomes on writing $\lambda = 2\ i\ \xi$ and using (7)

$$Q = -\frac{1}{2}\left[\sum_{k=1}^{n-1} (2 - \lambda\ a_{kk})\ t_k{}^2 + \sum_{j \neq k}^{n-1} (1 - \lambda\ a_{jk})\ t_j\ t_k\right], \tag{10}$$

which is no longer Hermitian, yet could be transformed into the standard form :

$$Q = -\frac{1}{2} \sum_{k=1}^{n-1} \Lambda_k\ t'^2_k, \tag{11}$$

where Λ's are the roots of the characteristic equation (12) below, and their real parts are all positive.[1]

$$\Delta_n\ (\lambda, \Lambda) = \begin{vmatrix} 2-\lambda\ a_{1,1}-\Lambda & 1-\lambda\ a_{1,2} & \cdots\cdots & 1-\lambda\ a_{1,n-1} \\ 1-\lambda\ a_{2,1} & 2-\lambda\ a_{2,2}-\Lambda & \cdots\cdots & 1-\lambda\ a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1-\lambda\ a_{n-1,1} & 1-\lambda_{n-1,2} & \cdots\cdots & 2-\lambda\ a_{n-1,n-1}-\Lambda \end{vmatrix} = 0, \tag{12}$$

where $a_{jk} = \dfrac{n-g}{n^2}$, and $g = \max\ (j, k)$.

1) The detail is reported by the author in Shikoku Sugaku Shijo Danwa (Japanese,) No. 3, 1951.

Substituting (11) in (9), the multiple integral decomposes into $n-1$ simple integrals, and each of them becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}\Lambda_k t'^2_k\right)dt'_k = 1/\Lambda_k^{\frac{1}{2}}\ (k=1, 2, \cdots\cdots, n-1).$$

Hence (9) may be written

$$\psi_n(\xi)=\sqrt{n/\Lambda_1\Lambda_2\cdots\cdots\Lambda_{n-1}}. \tag{13}$$

Here the denominator being the product of all roots of the equation (12), it is nothing but the absolute term : namely

$$\Delta_n(\lambda, 0) = \begin{vmatrix} 2-\frac{n-1}{n^2}\lambda & 1-\frac{n-2}{n^2}\lambda & \cdots\cdots\cdots & 1-\frac{\lambda}{n^2} \\ 1-\frac{n-2}{n^2}\lambda & 2-\frac{n-2}{n^2}\lambda & \cdots\cdots\cdots & 1-\frac{\lambda}{n^2} \\ \cdots & \cdots & \cdots & \cdots \\ 1-\frac{\lambda}{n^2} & 1-\frac{\lambda}{n^2} & \cdots\cdots\cdots & 2-\frac{\lambda}{n^2} \end{vmatrix}, \tag{14}$$

the expansion of which yields

$$\frac{\Delta_n(\lambda,0)}{n}\equiv D_n(\lambda)=1-\frac{\lambda}{3\,!}\left(1-\frac{1}{n^2}\right)+\frac{\lambda^2}{5\,!}\left(1-\frac{1}{n^2}\right)\left(1-\frac{4}{n^2}\right)$$

$$-\frac{\lambda^3}{7\,!}\left(1-\frac{1}{n^2}\right)\left(1-\frac{4}{n^2}\right)\left(1-\frac{9}{n^2}\right)+\cdots\cdots+\frac{(-\lambda)^{n-1}}{(2n-1)\,!}\prod_{\nu=1}^{n-1}\left(1-\frac{\nu^2}{n^2}\right). \tag{15}$$

Or putting $\lambda=n^2\zeta$, we obtain polynomial of degree $n-1$ in ζ :

$$(-1)^{n-1} n\ D_n(\lambda)=\sum_{l=0}^{n-1}(-1)^l\binom{2n-1-l}{l}\zeta^{n-1-l}\equiv P_{n-1}(\zeta). \tag{16}$$

Since the roots of the equation $P_{n-1}(\zeta)=0$ are all real positive and different, we may arrange them in ascending order

$$0<\zeta_1<\zeta_2<\cdots\cdots<\zeta_{n-1},$$

and write $P_{n-1}(\zeta)=\prod_{\nu=1}^{n-1}(\zeta-\zeta_\nu)$.

The actual forms of the polynomials (16) for $n=2, 3, \cdots\cdots$ are given in the following

Table I

n	$P_{n-1}(\zeta)=\zeta^{n-1}-2(n-1)\zeta^{n-2}+(n-2)(2n-3)\zeta^{n-3}-\cdots\cdots+(-1)^{n-1}n$
2	$P_1=\zeta-2$
3	$P_2=\zeta^2-4\zeta+3=(\zeta-1)(\zeta-3)=X-1$, where $X=(\zeta-2)^2$
4	$P_3=(\zeta-2)(\zeta-2+\sqrt{2})(\zeta-2-\sqrt{2})$
5	$P_4=X^2-3X+1$, where $X=(\zeta-2)^2$

6	$P_5=(\zeta-1)(\zeta-2)(\zeta-3)(\zeta-2+\sqrt{3})(\zeta-2-\sqrt{3})$
7	$P_6=X^3-5X^2+6X-1$, $X=(\zeta-2)^2$; the 3 roots of $P_6(X)=0$ lie in $\langle 0, 1\rangle$, $\langle 1, 2\rangle$ and $\langle 3, 4\rangle$.
8	$P_7=(\zeta-2)(\zeta^6-12\zeta^5+54\zeta^4-112\zeta^3+106\zeta^2-40\zeta+4)$. The roots $(\neq 2)$ of $P_7=0$ lie two in $\langle 0, 1\rangle$, one in $\langle 1, 2\rangle$ as well as $\langle 2. 3\rangle$, and two in $\langle 3, 4\rangle$
9	$P_8=(X-1)(X^3-6X^2+9X-1)$, where $X=(\zeta-2)^2$, so that besides $X=1$, the roots lie in $\langle 0, 1\rangle$, $\langle 2, 3\rangle$ and $\langle 3, 4\rangle$.
10	$P_9=(\zeta-2)(\zeta^8-16\zeta^7+104\zeta^6-352\zeta^5+661\zeta^4-680\zeta^3+356\zeta^2-80\zeta+5)$. Besides $\zeta=2$, there are three roots in $\langle 0, 1\rangle$ as well as $\langle 3, 4\rangle$, and one root in $\langle 1, 2\rangle$ as well as $\langle 2, 3\rangle$
11	$P_{10}=X^5-9X^4+28X^3-35X^2+15X-1$, $X=(\zeta-2)^2$. The roots X lie in $\langle 0, 1/2\rangle$, $\langle 1/2, 1\rangle$, $\langle 1, 2\rangle$, $\langle 2, 3\rangle$ and $\langle 3, 4\rangle$.
12	$P_{11}=(\zeta-1)(\zeta-2)(\zeta-3)f(\zeta)$, where $f(\zeta)=\zeta^8-16\zeta^7+103\zeta^6-340\zeta^5+607\zeta^4-568\zeta^3+251\zeta^2-44\zeta+2$. The number of variations in signs of Sturm functions of f at $\zeta=0, 1, 2, 3, 4$ is 8, 5, 4, 3, 0 respectively, so that $P_{11}=0$ has just eleven positive roots.
	.........................., and so on.

From all the foregoing we obtain the required characteristic

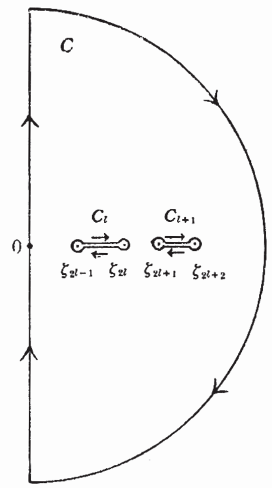
$$\psi_n(\xi)=\sqrt{\frac{n}{\Delta_n(\lambda, 0)}}=\frac{1}{\sqrt{D_n(\lambda)}}=\sqrt{\frac{(-1)^{n-1}n}{P_{n-1}(\zeta)}}, \tag{17}$$

where $\lambda=2i\xi=n^2\zeta$. Correspondingly we get the probability density function, in view of (7) (8) and (9),

$$\varphi(\omega_n^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\xi\omega_n^2)\,\psi_n(\xi)\,d\xi=\frac{1}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{\lambda}{2}\omega_n^2\right)}{\sqrt{-D_n(\lambda)}}d\lambda$$

$$=\frac{n^{\frac{5}{2}}}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{n^2}{2}\omega_n^2\,\xi\right)}{\sqrt{(-1)^n P_{n-1}(\zeta)}}d\zeta. \tag{18}$$

In the last integral the path of integration is the whole imaginary axis in the ζ-plane. But, since the integrand is analytic in ζ, and moreover the integral taken along the right semi-circle drawn in the plane with the origin as centre, tends to 0, as radius $\longrightarrow\infty$, so we may conceive the path of integration in (18) as one contour, composed of the semi-circle together with the imaginary axis. Denote it by C. Furthermore set several barriers along each segment joining successively, two by two, branch points $\zeta_1, \zeta_2, \ldots\ldots, \zeta_{n-1}$ (besides, if n be even, the ∞-point shall be taken as ζ_n), and thus we obtain $\overrightarrow{\zeta_1 \zeta_2}$, $\overrightarrow{\zeta_3 \zeta_4}, \ldots\ldots$ Describe around them small contours C_l ($l=1, 2, \ldots\ldots$ to $\frac{n-1}{2}$ or $\frac{n}{2}$ according as $n=$odd or even). By Cauchy's theorem, the integral taken along C can be replaced by those taken along $\sum C_l$, or what is the same thing as taken twice along every segment $\zeta_{2l-1}\zeta_{2l}$ with the sign alternately changed In fact, the sign of the square-root in (18) should change after one complete revolution about every branch point ζ_k, so that in contour C_l the integral taken along $\overrightarrow{\zeta_{2l-1}\zeta_{2l}}$ is equal to that taken along $\overrightarrow{\zeta_{2l}\zeta_{2l-1}}$ after one revolution about ζ_{2l}, and thus the result shall be doubled. Of course, the infinitely small circle about ζ_k contributes nothing. On the otherhand, if each half revolution about ζ_{2l} and ζ_{2l+1} be done, the sign changes, and accordingly the integrals round C_l and C_{l+1} must have the oppotite sign. From all these, we have



$$\varphi(\omega_n^2) = \frac{n^{\frac{5}{2}}}{2\pi} \sum_{k=1}^{p} (-1)^{k-1} \int_{\zeta_{2k-1}}^{\zeta_{2k}} \frac{\exp\left(-\frac{1}{2} n^2 \omega_n^2 \zeta\right)}{\sqrt{(-1)^n P_{n-1}(\zeta)}} d\zeta, \tag{19}$$

where $p = \frac{n-1}{2}$ or $\frac{n}{2}$ according as $n=$odd or even, and in the latter case we make $\zeta_n = \infty$.

Now integrating (19) with regards to ω_n^2 from ω_n^2 to ∞, we get

$$\int_{\omega_n^2}^{\infty} \varphi(\omega_n^2)\, d\omega_n^2 = 1 - \Phi(\omega_n^2) = \frac{\sqrt{n}}{\pi} \int_{\zeta_{2k-1}}^{\zeta_{2k}} \frac{\exp\left(-\frac{1}{2} n^2 \omega_n^2 \zeta\right)\, d\zeta}{\sqrt{(-1)^n P_{n-1}(\zeta)} \quad \zeta}, \tag{20}$$

For example, we have in case $n=2$ (cf Table I)

$$1 - \Phi(\omega_2^2) = \frac{\sqrt{2}}{\pi} \int_2^{\infty} \frac{\exp(-2\omega_2^2 \zeta)}{\zeta\sqrt{\zeta-2}} d\zeta = \frac{2}{\pi} \int_0^{\pi/2} \exp(-4\omega_2^2 \sec^2\theta)\, d\theta.$$

So

$$\left.\begin{aligned} \varphi(\omega_2^2) &= \Phi'(\omega_2^2) = \frac{8}{\pi} \int_0^{\pi/2} \exp(-4\omega_2^2 \sec^2\theta).\ \sec^2\theta\, d\theta, \\ \varphi(0) &= \Phi'(0) = \frac{8}{\pi} \int_0^{\pi/2} \sec^2\theta\, d\theta = \infty, \end{aligned}\right\} \tag{21}$$

and thus the frequency curve for ω_2^2 becomes J-shaped.

§ **3.** *The* ω_∞^2 *distribution*. Now we can pass into Smirnoff's ω^2 by making $n \to \infty$ in (15) and (17) : thus

$$\lim_{n\to\infty}\frac{\Delta_n(\lambda,0)}{n}=1-\frac{\lambda}{3!}+\frac{\lambda^2}{5!}-\frac{\lambda^3}{7!}+\ldots\equiv\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\equiv D(\lambda)\ ;$$

$$\lim_{n\to\infty}\psi_n(\xi)\equiv\psi(\xi)=\frac{1}{\sqrt{D(\lambda)}}=\sqrt{\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}}}.$$

Hence $$\varphi(\omega^2)=\frac{1}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{\lambda}{2}\omega^2\right)}{\sqrt{-\sin\sqrt{\lambda}/\sqrt{\lambda}}}\,d\lambda,$$

and $$\int_{\omega 2}^{\infty}\varphi(\omega^2)\,d\,\omega^2=\frac{1}{2\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{1}{2}\omega^2\lambda\right)}{\sqrt{-\sin\sqrt{\lambda}/\sqrt{\lambda}}}\frac{d\lambda}{\lambda}. \tag{22}$$

Upon writing $\lambda=z^2$ and expressing the infinite integral as contour-integrals in exactly the same way as before, we get

$$1-\Phi(\omega^2)=\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(2k-1)\pi}^{2k\pi}\frac{\exp\left(-\frac{1}{2}\omega^2 z^2\right)}{\sqrt{-z\ \sin\ z}}\,dz. \tag{23}$$

It may be noticed that the integrand of (23) becomes only integrably infinite at $z=l\pi$ $(l=1,2,\ldots\ldots)$, and the same holds for its derivative :

$$\varphi(\omega^2)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(k-1)\pi}^{2k\pi}\frac{\exp\left(-\frac{1}{2}\omega^2 z^2\right)}{\sqrt{-\sin\ z}}z^{\frac{3}{2}}\,dz,$$

so that

$$\varphi(0)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(2k-1)\pi}^{2k\pi}\frac{z^{\frac{3}{2}}\,dz}{\sqrt{-\sin\ z}}=\pi^{\frac{3}{2}}\int_0^1\frac{du}{\sqrt{\sin\pi u}}\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1+u)^{\frac{3}{2}}. \tag{24}$$

Since the last series is summable in Cesàro-Hölder's sense, $\varphi(0)$ surely exists. Hence the frequency curve for $\omega_\infty{}^2$ is usual bell-shaped (cf (21)).

For the purpose of numerical computations, the very form (23) is inconvenient. We may, therefore, transform firstly

$$z=\left(2k-\frac{1}{2}\right)\pi+x,\qquad\left(-\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2}\right),$$

and secondly $$\sqrt{-\sin z}=\sqrt{\sin\left(\frac{\pi}{2}-x\right)}=\sqrt{\cos x}=\cos\frac{\pi}{2}t,$$

$$\cos z\,dz=\pi\cos\frac{\pi}{2}t\ \sin\frac{\pi}{2}t\,dt\qquad(-1\leqq t\leqq 1).$$

Here, by means of the relation $\cos^2\frac{\pi}{2}t=\cos x$, we can put the points of the intervals $-1\leqq t\leqq 1$ and $-\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2}$ in one to one correspondence with each other, so that the value $z=\left(2k-\frac{1}{2}\right)\pi+x$ shall be determined uniquely when t is assigned. Thus the integral

(23) may be written as

$$1-\Phi(\omega^2)=2\sum_{k=1}^{\infty}(-1)^{k-1}\int_{-1}^{1}\frac{\exp\left(-\frac{1}{2}\omega^2 z^2\right)}{\sqrt{z}}\cdot\frac{dt}{\sqrt{1+\cos^2\frac{\pi}{2}t}}$$

$$=4\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2}\int_{-1}^{1}f(t)\,dt. \tag{25}$$

To evaluate this integral, we may utilize e. g. the so-called Gauss' method of 5 selected ordinates :

$$\frac{1}{2}\int_{-1}^{1}f(t)\,dt=\sum_{\nu=1}^{5}R_\nu\, y_\nu.$$

On calculating the value $y_\nu=f(t_\nu, \omega,^2 k)$ for every t_ν $(\nu=1, 2, 3, 4, 5)$, we can readily compute

$$1-\Phi(\omega^2)=4\sum_{k=1}^{\infty}(-1)^{k-1}\sum_{\nu=1}^{5}R_\nu\, y_\nu(t_\nu, \omega,^2 k).$$

For large values of ω^2, the convergence is quite rapid, in favour of the exponential factor with negative index, and besides the series being alternate, the calculation might be stopped as the summand becomes small enough. The Table of ω^2 distribution thus obtained by Y. Ueda, l. c., is reprinted at the end of this note.

§ **4.** *Application.* Supposing an empirical frequency distribution, divide the whole interval $\langle a, b\rangle$ into n equal subintervals (classes), and let the points of divisions be $a=x_0$, $x_1, \ldots,$ $x_n=b$. If the empirically obtained number of individuals falling in $\langle x_{k-1}, x_k\rangle$ be m_k, then

$$\sum_{k=1}^{k} m_k=S(x_k)=NS_N(x_k),\quad k=1, 2, \ldots\ldots, N,$$

and $S(x)=0$ if $x<a$, while $S(x)=N$ if $x>b$.
Hence by definition (2) we have

$$\omega^2=\frac{1}{N}\sum_{k=0}^{n+1}\int_{x_{k-1}}^{x_k}\Big[S(x)-NF(x)\Big]^2 dF,\quad \text{where}\quad x_{-1}=-\infty,\quad x_{n+1}=\infty,$$

which can be written

$$\omega^2=N\int_{-\infty}^{a}F(x)^2\,dF+\frac{1}{N}\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\Big[S(x)-NF(x)\Big]^2 dF+N\int_{b}^{\infty}(1-F(x))^2\,dF$$

$$=(\text{i})+(\text{ii})+(\text{iii}),$$

Here $\quad (\text{i})=\Big[\frac{1}{3}NF(x)^3\Big]_{-\infty}^{a}=\frac{1}{3}NF(a)^3=0\ (\text{or}\ \fallingdotseq 0)$

and $\quad (\text{iii})=\frac{1}{3}\Big[N(1-F(x))^3\Big]_{b}^{\infty}=-\frac{1}{3}N(1-F(b))^3=0\ (\text{or}\ \fallingdotseq 0),$

so that (ii) is only preponderant. However, since the values of $S(x)$ are determined merely at the points of divisions, we must be contented with approximation obtained by summation. E. g. using the trapezoidal formula

$$\omega^2 = \frac{h}{N} \sum_{k=1}^{n} \Big[S(x_k) - NF(x_k) \Big]^2 F'(x_k), \tag{26}$$

where $h=(b-a)/n$ is the width of one subinterval. (Strictly speaking, the values for end subintervals shall be multiplied by 1/2, but as they are small, it is immaterial).

Example. The frequency distribution of stature for 1000 (which is a relative frequency, the true number being 629779) males of aged 20 in Japan at a certain year is given in Table Ⅱ. Upon applying Pearson's method of moments, the frequency distibution is fitted by the normal curve :

$$y = f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-a)^2}{2\sigma^2} \right\}, \quad \text{with} \quad a = 160.285, \quad \sigma = 5.8426.$$

We have to test the goodness of fitting.

Previously transform the variable into $(x-a)/\sigma = t$, and referring to the normal probability table, find the values

$$F(t_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_k} \exp\left(-\frac{1}{2}t^2\right) dt, \quad F'(t_k) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t_k^2\right).$$

On evaluating each summand in (26), as in Table Ⅱ, we obtain

Table Ⅱ

x_k (cm)	m_k	$S(x)$	$t_k=(x_k-a)/\sigma$	$NF(t_k)$	$(S-NF)^2 F'$
135	0	0	−4.328	0.01	0.0000
140	1	1	−3.472	0.26	0.0005
145	5	6	−2.616	4.45	0.0313
150	32	38	−1.7603	39.17	0.1160
155	141	179	−0.9046	182.84	3.9024
160	300	479	−0.0488	480.55	0.9696
165	316	795	+0.8070	790.16	6.7480
170	158	953	1.6628	951.84	0 1347
175	40	993	2.519	994.11	0 0206
180	6	999	3.374	999.63	0.0005
185	1	1000	4.230	999.99	0.0000
	N = 1000			total	11.9236

$$\omega^2 = \frac{1}{N} \times \frac{50}{100} \times 11{\cdot}9236 = 0.0102.$$

From the annexed Table of $\varPhi(\omega^2)$, we find $\varPhi(0.0102)=0.0007$, so that $1-\varPhi=0.9993>0.05$. Thus the ω^2—test docs not reject the hypothetical distribution.

For me, however, it seems somewhat improper to test the above by the $\omega_\infty{}^2$ distribution. If we could prepare a table of $\omega_{10}{}^2$ distribution, probably the value of $\varPhi(\omega_{10}{}^2=0.0102)$ would be $\gg \varPhi(\omega_\infty{}^2=0.0102)=0.0007$, as may be guessed from the fact that $\varPhi'(\omega_2^2=0)=\infty$, whereas $\varPhi'(\omega_\infty{}^2=0)$ is finite (cf (21) and (24)).——Moreover, from $\varPhi(\omega^2)$—Table, it appears seemingly that $\varPhi'(0)=0$. At least, we can assert that $\varPhi'(0) < 0.0001/0.01=0.01$, because $\varPhi''(0) \fallingdotseq [\varPhi(0.02)-2\varPhi(0.01)+\varPhi(0)]/0.01^2 > 0$.

Table of $\varPhi(\omega^2) = \int_0^{\omega^2} \varphi(\omega^2)\, d\omega^2$

ω^2	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	$\varPhi=0$	.0001	.0030	.0240	.0633	.1240	.1863	.2486	.3084	.3641
0.1	.4154	.4622	.5047	.5435	.5787	.6107	.6397	.6662	.6904	.7125
0.2	.7338	.7513	.7684	.7840	.7984	.8118	.8242	.8356	.8461	.8559
0.3	.8650	.8735	.8813	.8887	.8955	.9018	.9093	.9133	.9185	.9234
0.4	.9279	.9322	.9361	.9399	.9434	.9467	.9497	.9526	.9555	.9579
0.5	.9603	.9626	.9647	.9667	.9686	.9703	.9720	.9736	.9751	.9764
0.6	.9777	.9790	.9801	.9812	.9823	.9832	.9842	.9850	.9858	.9866
0.7	.9875	.9880	.9887	.9893	.9899	.9904	.9910	.9914	.9919	.9923
0.8	.9928	.9931	.9935	.9939	.9942	.9945	.9948	.9951	.9953	.9956
0.9	.9958	.9960	.9962	.9965	.9966	.9968	.9970	.9971	.9973	.9974

ω^2	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	∞
$\varPhi$	.9976	.9985	.9991	.9995	.9997	.9998	.9999	.9999	1.0000	1

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