On the ω^2 Distribution

By

Yoshikatsu WATANABE

(Received Sept. 30, 1951)

As a criterion for testing the goodness of fit of frequency function etc., hitherto K. Pearson's χ^2 test was extensively applied. However, in the quantity

$$\chi^2 = \Sigma w_\nu (m_\nu - Np_\nu)^2,$$

the weight being assumed to be $w_\nu = 1/Np_\nu$, those data with less probabilities have unreasonably more effect. On the contrary, the ω^2 test, originally due to R.v. Mises and H. Cramér and improved by N. Smirnoff[1)] is quite free from this defect. Nevertheless its table seems not yet to have been found. For this requirement, Y. Ueda constructed a table of ω^2_∞ distribution, as his graduation thesis at Waseda University, under the direction of the author. But it appears to be rather desirable to prepare those of $\omega_n{}^2$ distributions, n being the number of classes in any statistics, say 5∼25. In the present note, however, it is only theoretically developped how to obtain the ω_n distribution parallel to Smirnoff's ω^2_∞ distribution, while the actual numerical computations as to ω_n are about going to be executed by few students in our institute, and some results might be expected.

§ 1. *Definitions.* Let some random variable x be subject to the probability density function $f(x)$, and the cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(x)\, dx.$$

Let $x_1 \leqq x_2 \leqq \cdots\cdots \leqq x_N$ be the observed values of x in N experimental trials, and

$$S(x) = NS_N(x)$$

denote the number of individuals which do not exceed a given value x, so that $S_N(x) = S(x)/N$ is an empirical accumlated probability, and increases in a stairway. Accordingly

$$\delta(x) = NS_N(x) - NF(x)$$

gives the deviation at $x = x$ of the experimental value from the theoretical. Now we define

$$\omega^2 = \frac{1}{N}\int_{-\infty}^{\infty} w(x)\ \delta(x)^2\, dx = N\int_{-\infty}^{\infty} w(x) \Big[S_N(x) - F(x)\Big]^2 dx, \tag{1}$$

where $w(x)$ is the weight. Thus ω^2 affords the degree of deviation, and the smaller it is, the more precisely the assumed function represents the actual feature (R. v. Mises). Further N.

1) N. Smirnoff : Sur la distribution de ω^2, Comptes Rendus, 202 (1936), p. 449.

Smirnoff adopted the following form :

$$\omega^2 = N\int_{-\infty}^{\infty} w(F(x))\left[S_N(x)-F(x)\right]^2 F'(x)\,dx, \tag{2}$$

and in particular, taking the weight $w(F(x))=1$,

$$\omega^2 = N\int_{-\infty}^{\infty} \left[S_N(x)-F(x)\right]^2 f(x)\,dx, \qquad (f(x)=F'(x)). \tag{3}$$

The last form can be obtained by putting $w(x)=f(x)$ in (1), and thus the weight being directly propotional to the frequency $f(x)$, it is more legitimate than χ^2, in which the weight is inversely proportional to $f(x)$. Furthermore, while v. Mises' original ω^2 is subjected to the selection of $F(x)$, Smirnoff's ω^2, as Stieltjes integral, is quite independent of it, and therefore could be evenly applied for any distribution.

§ **2.** *The distribution function of* ω^2_n, $\Phi(\omega^2_n)$. Smirnoff's ω^2 being independent of the choice of F, we may after him assume the case of the uniform distribution in $\langle 0, 1\rangle$, so that

$$\begin{aligned} f(x)&=0, & F(x)&=0, &&\text{for } x<0,\\ &=1, & &=x, &&\text{for } 0\leqq x\leqq 1,\\ &=0, & &=1, &&\text{for } x>1. \end{aligned}$$

Evidently the theoretical probability that an observed point falls in the partial interval

$$\frac{k-1}{n}\leqq x\leqq\frac{k}{n} \qquad (k=1,\,2,\cdots\cdots,\,n) \tag{4}$$

is $p_k=1/n$, and the cumulative probability that x does not exceed l/n is l/n. Hence, if m_k be the empirical number of points having fallen in the interval (4) in N trials, then Pearson's weighted deviation[1] becomes

$$t_k=\left(m_k-Np_k\right)/\sqrt{Np_k}=\left(m_k-\frac{N}{n}\right)/\sqrt{\frac{N}{n}}, \tag{5}$$

where $k=1,\,2,\cdots\cdots,\,n-1$, and $t_n=-\sum_{\nu=1}^{n-1}t_\nu$.

If N be sufficiently great, and n tolerably large, say $n=20$, the integral (2) may be approximated by summation concerning n intervals (4) as follows :

$$\omega^2_{N,n}=\frac{N}{n}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\left[\frac{m_1+\cdots\cdots+m_l}{N}-\frac{l}{n}\right]^2=\frac{1}{nN}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\left[\sum_{k=1}^{l}\left(m_k-\frac{N}{n}\right)\right]^2$$

$$=\frac{1}{nN}\sum_{l=1}^{n-1}w\left(\frac{l}{n}\right)\frac{N}{n}\left(\sum_{k=1}^{l}t_k\right)^2=\frac{1}{n^2}\sum_{j,k}^{n-1}t_j\,t_k\sum_{l=j}^{n-1}w\left(\frac{l}{n}\right), \tag{6}$$

1) As we have remarked before, Pearson's weight is inadequate, yet we may utilize it merely for the sake of convenient transformation.

where $g = \text{Max}\ (j, k)$. That is

$$\omega^2_{N,n} = \sum_{j,k=1}^{n-1} a_{jk}\ t_j\ t_k, \quad \text{where} \quad a_{jk} = a_{kj} = \frac{1}{n^2} \sum_{l=g}^{n-1} w\left(\frac{l}{n}\right).$$

Specially for $w = 1$, we have $a_{jk} = a_{kj} = \dfrac{n-g}{n^2}$, and therefore

$$\omega^2_{N,n} = \sum_{j,k=1}^{n-1} \frac{n-g}{n^2} t_j\ t_k = \sum a_{jk}\ t_j\ t_k = A\ (t,\ t). \tag{7}$$

Thus our ω^2 is reduced to a positive definite Hermite form.

Now we require to find the c. d. f. $\Phi_{N,n}\ (\omega^2)$, or its p. d. f. $\varphi\ (\omega^2)$. For this purpose, we shall begin to determine its characteristic

$$\psi_{N,n}\ (\xi) = \sum_{m_1 + \cdots + m_n = N} p\ (m_1, \cdots\cdots, m_n)\ \exp\left\{i\ \xi\ A\ (t, t)\right\}, \quad i = \sqrt{-1}, \tag{8}$$

where p denotes the probability that the respective number of points falling in the k-th subinterval becomes m_k $(k = 1, \cdots\cdots, n)$.

When $N \longrightarrow \infty$, as is well known in χ^2-distribution, the expression (8) shall tend to the limit :

$$\lim_{N \to \infty} \psi_{N,n}\ (\xi) \equiv \psi_n(\xi) = \frac{\sqrt{n}}{\sqrt{2\pi}^{\,n-1}} \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\chi^2 + i\ \xi\ A\ (t, t)\right\} dt_1 \cdots dt_{n-1}, \tag{9}$$

where $\chi^2 = \sum_{k=1}^{n} t_k{}^2 = \sum_{k=1}^{n-1} t_k{}^2 + \left(-\sum_{k=1}^{n-1} t_k\right)^2 = 2 \sum_{k=1}^{n-1} t_k{}^2 + \sum_{j \neq k}^{n-1} t_j\ t_k.$

Hence the exponent in (9) becomes on writing $\lambda = 2\ i\ \xi$ and using (7)

$$Q = -\frac{1}{2} \left[\sum_{k=1}^{n-1} (2 - \lambda\ a_{kk})\ t_k{}^2 + \sum_{j \neq k}^{n-1} (1 - \lambda\ a_{jk})\ t_j\ t_k\right], \tag{10}$$

which is no longer Hermitian, yet could be transformed into the standard form :

$$Q = -\frac{1}{2} \sum_{k=1}^{n-1} \Lambda_k\ t'^2_k, \tag{11}$$

where Λ's are the roots of the characteristic equation (12) below, and their real parts are all positive.[1]

$$\Delta_n\ (\lambda, \Lambda) = \begin{vmatrix} 2 - \lambda\ a_{1,1} - \Lambda & 1 - \lambda\ a_{1,2} & \cdots\cdots & 1 - \lambda\ a_{1,n-1} \\ 1 - \lambda\ a_{2,1} & 2 - \lambda\ a_{2,2} - \Lambda & \cdots\cdots & 1 - \lambda\ a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 - \lambda\ a_{n-1,1} & 1 - \lambda_{n-1,2} & \cdots\cdots & 2 - \lambda\ a_{n-1,n-1} - \Lambda \end{vmatrix} = 0, \tag{12}$$

where $a_{jk} = \dfrac{n-g}{n^2}$, and $g = \max\ (j, k)$.

---

1) The detail is reported by the author in Shikoku Sugaku Shijo Danwa (Japanese,) No. 3, 1951.

Substituting (11) in (9), the multiple integral decomposes into $n-1$ simple integrals, and each of them becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}\Lambda_k t'^2_k\right)dt'_k = 1/\Lambda_k^{\frac{1}{2}} \quad (k=1, 2, \cdots\cdots, n-1).$$

Hence (9) may be written

$$\psi_n(\xi)=\sqrt{n/\Lambda_1 \Lambda_2 \cdots\cdots \Lambda_{n-1}}. \tag{13}$$

Here the denominator being the product of all roots of the equation (12), it is nothing but the absolute term : namely

$$\Delta_n(\lambda, 0) = \begin{vmatrix} 2-\frac{n-1}{n^2}\lambda & 1-\frac{n-2}{n^2}\lambda & \cdots\cdots\cdots & 1-\frac{\lambda}{n^2} \\ 1-\frac{n-2}{n^2}\lambda & 2-\frac{n-2}{n^2}\lambda & \cdots\cdots\cdots & 1-\frac{\lambda}{n^2} \\ \cdots & \cdots & \cdots & \cdots \\ 1-\frac{\lambda}{n^2} & 1-\frac{\lambda}{n^2} & \cdots\cdots\cdots & 2-\frac{\lambda}{n^2} \end{vmatrix}, \tag{14}$$

the expansion of which yields

$$\frac{\Delta_n(\lambda, 0)}{n}\equiv D_n(\lambda)=1-\frac{\lambda}{3\,!}\left(1-\frac{1}{n^2}\right)+\frac{\lambda^2}{5\,!}\left(1-\frac{1}{n^2}\right)\left(1-\frac{4}{n^2}\right)$$

$$-\frac{\lambda^3}{7\,!}\left(1-\frac{1}{n^2}\right)\left(1-\frac{4}{n^2}\right)\left(1-\frac{9}{n^2}\right)+\cdots\cdots+\frac{(-\lambda)^{n-1}}{(2n-1)\,!}\prod_{\nu=1}^{n-1}\left(1-\frac{\nu^2}{n^2}\right). \tag{15}$$

Or putting $\lambda=n^2\zeta$, we obtain polynomial of degree $n-1$ in ζ :

$$(-1)^{n-1} n\, D_n(\lambda)=\sum_{l=0}^{n-1}(-1)^l \binom{2n-1-l}{l}\zeta^{n-1-l}\equiv P_{n-1}(\zeta). \tag{16}$$

Since the roots of the equation $P_{n-1}(\zeta)=0$ are all real positive and different, we may arrange them in ascending order

$$0<\zeta_1<\zeta_2<\cdots\cdots<\zeta_{n-1},$$

and write $P_{n-1}(\zeta)=\prod_{\nu=1}^{n-1}(\zeta-\zeta_\nu)$.

The actual forms of the polynomials (16) for $n=2, 3, \cdots\cdots$ are given in the following

Table I

n	$P_{n-1}(\zeta)=\zeta^{n-1}-2(n-1)\zeta^{n-2}+(n-2)(2n-3)\zeta^{n-3}-\cdots\cdots+(-1)^{n-1}n$
2	$P_1=\zeta-2$
3	$P_2=\zeta^2-4\zeta+3=(\zeta-1)(\zeta-3)=X-1$, where $X=(\zeta-2)^2$
4	$P_3=(\zeta-2)(\zeta-2+\sqrt{2})(\zeta-2-\sqrt{2})$
5	$P_4=X^2-3X+1$, where $X=(\zeta-2)^2$

6	$P_5=(\zeta-1)(\zeta-2)(\zeta-3)(\zeta-2+\sqrt{3})(\zeta-2-\sqrt{3})$
7	$P_6=X^3-5X^2+6X-1,\quad X=(\zeta-2)^2$; the 3 roots of $P_6(X)=0$ lie in $\langle 0, 1\rangle$, $\langle 1, 2\rangle$ and $\langle 3, 4\rangle$.
8	$P_7=(\zeta-2)(\zeta^6-12\zeta^5+54\zeta^4-112\zeta^3+106\zeta^2-40\zeta+4)$. The roots $(\neq 2)$ of $P_7=0$ lie two in $\langle 0, 1\rangle$, one in $\langle 1, 2\rangle$ as well as $\langle 2. 3\rangle$, and two in $\langle 3, 4\rangle$
9	$P_8=(X-1)(X^3-6X^2+9X-1)$, where $X=(\zeta-2)^2$, so that besides $X=1$, the roots lie in $\langle 0, 1\rangle$, $\langle 2, 3\rangle$ and $\langle 3, 4\rangle$.
10	$P_9=(\zeta-2)(\zeta^8-16\zeta^7+104\zeta^6-352\zeta^5+661\zeta^4-680\zeta^3+356\zeta^2-80\zeta+5)$. Besides $\zeta=2$, there are three roots in $\langle 0, 1\rangle$ as well as $\langle 3, 4\rangle$, and one root in $\langle 1, 2\rangle$ as well as $\langle 2, 3\rangle$
11	$P_{10}=X^5-9X^4+28X^3-35X^2+15X-1,\quad X=(\zeta-2)^2$. The roots X lie in $\langle 0, 1/2\rangle$, $\langle 1/2, 1\rangle$, $\langle 1, 2\rangle$, $\langle 2, 3\rangle$ and $\langle 3, 4\rangle$.
12	$P_{11}=(\zeta-1)(\zeta-2)(\zeta-3)f(\zeta)$, where $f(\zeta)=\zeta^8-16\zeta^7+103\zeta^6-340\zeta^5+607\zeta^4-568\zeta^3+251\zeta^2-44\zeta+2$. The number of variations in signs of Sturm functions of f at $\zeta=0, 1, 2, 3, 4$ is 8, 5, 4, 3, 0 respectively, so that $P_{11}=0$ has just eleven positive roots. . . . . , and so on.

From all the foregoing we obtain the required characteristic

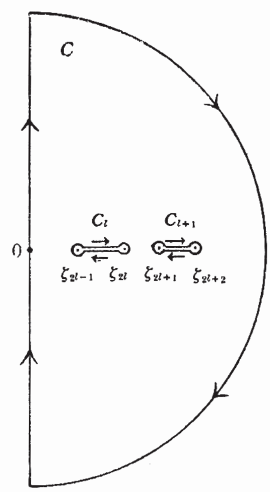
$$\psi_n(\xi)=\sqrt{\frac{n}{\Delta_n(\lambda, 0)}}=\frac{1}{\sqrt{D_n(\lambda)}}=\sqrt{\frac{(-1)^{n-1}n}{P_{n-1}(\zeta)}}, \tag{17}$$

where $\lambda=2i\xi=n^2\zeta$. Correspondingly we get the probability density function, in view of (7) (8) and (9),

$$\varphi(\omega_n^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\xi\omega_n^2)\,\psi_n(\xi)\,d\xi=\frac{1}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{\lambda}{2}\omega_n^2\right)}{\sqrt{-D_n(\lambda)}}d\lambda$$

$$=\frac{n^{\frac{5}{2}}}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{n^2}{2}\omega_n^2\,\xi\right)}{\sqrt{(-1)^n P_{n-1}(\zeta)}}d\zeta. \tag{18}$$

In the last integral the path of integration is the whole imaginary axis in the ζ-plane. But, since the integrand is analytic in ζ, and moreover the integral taken along the right semi-circle drawn in the plane with the origin as centre, tends to 0, as radius $\longrightarrow\infty$, so we may conceive the path of integration in (18) as one contour, composed of the semi-circle together with the imaginary axis. Denote it by C. Furthermore set several barriers along each segment joining successively, two by two, branch points $\zeta_1, \zeta_2, \ldots\ldots, \zeta_{n-1}$ (besides, if n be even, the ∞-point shall be taken as ζ_n), and thus we obtain $\overrightarrow{\zeta_1 \zeta_2}$, $\overrightarrow{\zeta_3 \zeta_4}, \ldots\ldots$ Describe around them small contours C_l ($l=1, 2, \ldots\ldots$ to $\frac{n-1}{2}$ or $\frac{n}{2}$ according as $n=$odd or even). By Cauchy's theorem, the integral taken along C can be replaced by those taken along $\sum C_l$, or what is the same thing as taken twice along every segment $\zeta_{2l-1}\zeta_{2l}$ with the sign alternately changed In fact, the sign of the square-root in (18) should change after one complete revolution about every branch point ζ_k, so that in contour C_l the integral taken along $\overrightarrow{\zeta_{2l-1} \zeta_{2l}}$ is equal to that taken along $\overrightarrow{\zeta_{2l} \zeta_{2l-1}}$ after one revolution about ζ_{2l}, and thus the result shall be doubled. Of course, the infinitely small circle about ζ_k contributes nothing. On the otherhand, if each half revolution about ζ_{2l} and ζ_{2l+1} be done, the sign changes, and accordingly the integrals round C_l and C_{l+1} must have the oppotite sign. From all these, we have



$$\varphi(\omega_n^2) = \frac{n^{\frac{5}{2}}}{2\pi} \sum_{k=1}^{p} (-1)^{k-1} \int_{\zeta_{2k-1}}^{\zeta_{2k}} \frac{\exp\left(-\frac{1}{2} n^2 \omega_n^2 \zeta\right)}{\sqrt{(-1)^n P_{n-1}(\zeta)}} d\zeta, \tag{19}$$

where $p = \frac{n-1}{2}$ or $\frac{n}{2}$ according as $n=$odd or even, and in the latter case we make $\zeta_n = \infty$.

Now integrating (19) with regards to ω_n^2 from ω_n^2 to ∞, we get

$$\int_{\omega_n^2}^{\infty} \varphi(\omega_n^2)\, d\omega_n^2 = 1 - \Phi(\omega_n^2) = \frac{\sqrt{n}}{\pi} \int_{\zeta_{2k-1}}^{\zeta_{2k}} \frac{\exp\left(-\frac{1}{2} n^2 \omega_n^2 \zeta\right)\, d\zeta}{\sqrt{(-1)^n P_{n-1}(\zeta)} \quad \zeta}, \tag{20}$$

For example, we have in case $n=2$ (cf Table I)

$$1 - \Phi(\omega_2^2) = \frac{\sqrt{2}}{\pi} \int_2^{\infty} \frac{\exp(-2\omega_2^2 \zeta)}{\zeta \sqrt{\zeta - 2}} d\zeta = \frac{2}{\pi} \int_0^{\pi/2} \exp(-4\omega_2^2 \sec^2 \theta)\, d\theta.$$

So

$$\left.\begin{aligned} \varphi(\omega_2^2) &= \Phi'(\omega_2^2) = \frac{8}{\pi} \int_0^{\pi/2} \exp(-4\omega_2^2 \sec^2 \theta).\ \sec^2 \theta\, d\theta, \\ \varphi(0) &= \Phi'(0) = \frac{8}{\pi} \int_0^{\pi/2} \sec^2 \theta\, d\theta = \infty, \end{aligned}\right\} \tag{21}$$

and thus the frequency curve for ω_2^2 becomes J-shaped.

§ **3.** *The ω_∞^2 distribution.* Now we can pass into Smirnoff's ω^2 by making $n \to \infty$ in (15) and (17) : thus

$$\lim_{n\to\infty}\frac{\Delta_n(\lambda,0)}{n}=1-\frac{\lambda}{3!}+\frac{\lambda^2}{5!}-\frac{\lambda^3}{7!}+\dots\equiv\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\equiv D(\lambda)\ ;$$

$$\lim_{n\to\infty}\psi_n(\xi)\equiv\psi(\xi)=\frac{1}{\sqrt{D(\lambda)}}=\sqrt{\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}}}.$$

Hence $$\varphi(\omega^2)=\frac{1}{4\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{\lambda}{2}\omega^2\right)}{\sqrt{-\sin\sqrt{\lambda}/\sqrt{\lambda}}}\,d\lambda,$$

and $$\int_{\omega 2}^{\infty}\varphi(\omega^2)\,d\omega^2=\frac{1}{2\pi}\int_{-\infty i}^{\infty i}\frac{\exp\left(-\frac{1}{2}\omega^2\lambda\right)}{\sqrt{-\sin\sqrt{\lambda}/\sqrt{\lambda}}}\frac{d\lambda}{\lambda}. \qquad (22)$$

Upon writing $\lambda=z^2$ and expressing the infinite integral as contour-integrals in exactly the same way as before, we get

$$1-\Phi(\omega^2)=\frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(2k-1)\pi}^{2k\pi}\frac{\exp\left(-\frac{1}{2}\omega^2z^2\right)}{\sqrt{-z\ \sin\ z}}\,dz. \qquad (23)$$

It may be noticed that the integrand of (23) becomes only integrably infinite at $z=l\pi$ $(l=1,2,\dots\dots)$, and the same holds for its derivative :

$$\varphi(\omega^2)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(k-1)\pi}^{2k\pi}\frac{\exp\left(-\frac{1}{2}\omega^2z^2\right)}{\sqrt{-\sin\ z}}\,z^{\frac{3}{2}}\,dz,$$

so that

$$\varphi(0)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\int_{(2k-1)\pi}^{2k\pi}\frac{z^{\frac{3}{2}}\,dz}{\sqrt{-\sin\ z}}=\pi^{\frac{3}{2}}\int_0^1\frac{du}{\sqrt{\sin\pi u}}\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1+u)^{\frac{3}{2}}. \qquad (24)$$

Since the last series is summable in Cesàro-Hölder's sense, $\varphi(0)$ surely exists. Hence the frequency curve for $\omega_\infty{}^2$ is usual bell-shaped (cf (21)).

For the purpose of numerical computations, the very form (23) is inconvenient. We may, therefore, transform firstly

$$z=\left(2k-\frac{1}{2}\right)\pi+x,\qquad\left(-\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2}\right),$$

and secondly $$\sqrt{-\sin z}=\sqrt{\sin\left(\frac{\pi}{2}-x\right)}=\sqrt{\cos x}=\cos\frac{\pi}{2}t,$$

$$\cos z\,dz=\pi\cos\frac{\pi}{2}t\ \sin\frac{\pi}{2}t\,dt\qquad(-1\leqq t\leqq 1).$$

Here, by means of the relation $\cos^2\frac{\pi}{2}t=\cos x$, we can put the points of the intervals $-1\leqq t\leqq 1$ and $-\frac{\pi}{2}\leqq x\leqq\frac{\pi}{2}$ in one to one correspondence with each other, so that the value $z=\left(2k-\frac{1}{2}\right)\pi+x$ shall be determined uniquely when t is assigned. Thus the integral

(23) may be written as

$$1-\Phi(\omega^2)=2\sum_{k=1}^{\infty}(-1)^{k-1}\int_{-1}^{1}\frac{\exp\left(-\frac{1}{2}\omega^2 z^2\right)}{\sqrt{z}}\cdot\frac{dt}{\sqrt{1+\cos^2\frac{\pi}{2}t}}$$

$$=4\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2}\int_{-1}^{1}f(t)\,dt. \tag{25}$$

To evaluate this integral, we may utilize e. g. the so-called Gauss' method of 5 selected ordinates :

$$\frac{1}{2}\int_{-1}^{1}f(t)\,dt=\sum_{\nu=1}^{5}R_\nu\, y_\nu\,.$$

On calculating the value $y_\nu=f(t_\nu,\omega,^2 k)$ for every t_ν $(\nu=1,2,3,4,5)$, we can readily compute

$$1-\Phi(\omega^2)=4\sum_{k=1}^{\infty}(-1)^{k-1}\sum_{\nu=1}^{5}R_\nu\, y_\nu(t_\nu,\omega,^2 k)\,.$$

For large values of ω^2, the convergence is quite rapid, in favour of the exponential factor with negative index, and besides the series being alternate, the calculation might be stopped as the summand becomes small enough. The Table of ω^2 distribution thus obtained by Y. Ueda, l. c., is reprinted at the end of this note.

§ **4.** *Application.* Supposing an empirical frequency distribution, divide the whole interval $\langle a,b\rangle$ into n equal subintervals (classes), and let the points of divisions be $a=x_0,\ x_1,\ldots,$ $x_n=b$. If the empirically obtained number of individuals falling in $\langle x_{k-1},\ x_k\rangle$ be m_k, then

$$\sum_{k=1}^{k} m_k=S(x_k)=NS_N(x_k),\quad k=1,2,\ldots\ldots,\ N,$$

and $S(x)=0$ if $x<a$, while $S(x)=N$ if $x>b$.
Hence by definition (2) we have

$$\omega^2=\frac{1}{N}\sum_{k=0}^{n+1}\int_{x_{k-1}}^{x_k}\Big[S(x)-NF(x)\Big]^2 dF,\quad \text{where}\quad x_{-1}=-\infty,\quad x_{n+1}=\infty,$$

which can be written

$$\omega^2=N\int_{-\infty}^{a}F(x)^2\,dF+\frac{1}{N}\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}\Big[S(x)-NF(x)\Big]^2 dF+N\int_{b}^{\infty}(1-F(x))^2\,dF$$

$$=(\text{i})+(\text{ii})+(\text{iii}),$$

Here $\quad(\text{i})=\Big[\frac{1}{3}NF(x)^3\Big]_{-\infty}^{a}=\frac{1}{3}NF(a)^3=0\ (\text{or}\doteqdot 0)$

and $\quad(\text{iii})=\frac{1}{3}\Big[N(1-F(x))^3\Big]_{b}^{\infty}=-\frac{1}{3}N(1-F(b))^3=0\ (\text{or}\doteqdot 0),$

so that (ii) is only preponderant. However, since the values of $S(x)$ are determined merely at the points of divisions, we must be contented with approximation obtained by summation. E. g. using the trapezoidal formula

$$\omega^2 = \frac{h}{N} \sum_{k=1}^{n} \Big[S(x_k) - NF(x_k) \Big]^2 F'(x_k), \tag{26}$$

where $h=(b-a)/n$ is the width of one subinterval. (Strictly speaking, the values for end subintervals shall be multiplied by 1/2, but as they are small, it is immaterial).

Example. The frequency distribution of stature for 1000 (which is a relative frequency, the true number being 629779) males of aged 20 in Japan at a certain year is given in Table Ⅱ. Upon applying Pearson's method of moments, the frequency distibution is fitted by the normal curve :

$$y = f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}, \quad \text{with} \quad a = 160.285, \quad \sigma = 5.8426.$$

We have to test the goodness of fitting.

Previously transform the variable into $(x-a)/\sigma = t$, and referring to the normal probability table, find the values

$$F(t_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_k} \exp\left(-\frac{1}{2}t^2\right) dt, \quad F'(t_k) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t_k^2\right).$$

On evaluating each summand in (26), as in Table Ⅱ, we obtain

Table Ⅱ

x_k (cm)	m_k	$S(x)$	$t_k=(x_k-a)/\sigma$	$NF(t_k)$	$(S-NF)^2F'$
135	0	0	−4.328	0.01	0.0000
140	1	1	−3.472	0.26	0.0005
145	5	6	−2.616	4.45	0.0313
150	32	38	−1.7603	39.17	0.1160
155	141	179	−0.9046	182.84	3.9024
160	300	479	−0.0488	480.55	0.9696
165	316	795	+0.8070	790.16	6.7480
170	158	953	1.6628	951.84	0 1347
175	40	993	2.519	994.11	0 0206
180	6	999	3.374	999.63	0.0005
185	1	1000	4.230	999.99	0.0000
	N = 1000			total	11.9236

$$\omega^2 = \frac{1}{N} \times \frac{50}{100} \times 11{\cdot}9236 = 0.0102.$$

From the annexed Table of $\Phi(\omega^2)$, we find $\Phi(0.0102) = 0.0007$, so that $1 - \Phi = 0.9993 > 0.05$. Thus the ω^2—test docs not reject the hypothetical distribution.

For me, however, it seems somewhat improper to test the above by the $\omega_\infty{}^2$ distribution. If we could prepare a table of $\omega_{10}{}^2$ distribution, probably the value of $\Phi(\omega_{10}{}^2 = 0.0102)$ would be $\gg$ $\Phi(\omega_\infty{}^2 = 0.0102) = 0.0007$, as may be guessed from the fact that $\Phi'(\omega_2^2 = 0) = \infty$, whereas $\Phi'(\omega_\infty{}^2 = 0)$ is finite (cf (21) and (24)).——Moreover, from $\Phi(\omega^2)$—Table, it appears seemingly that $\Phi'(0) = 0$. At least, we can assert that $\Phi'(0) < 0.0001/0.01 = 0.01$, because $\Phi''(0) \fallingdotseq [\Phi(0.02) - 2\,\Phi(0.01) + \Phi(0)]/0.01^2 > 0$.

Table of $\Phi(\omega^2) = \int_0^{\omega^2} \varphi(\omega^2)\, d\,\omega^2$

ω^2	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	$\Phi = 0$	.0001	.0030	.0240	.0633	.1240	.1863	.2486	.3084	.3641
0.1	.4154	.4622	.5047	.5435	.5787	.6107	.6397	.6662	.6904	.7125
0.2	.7338	.7513	.7684	.7840	.7984	.8118	.8242	.8356	.8461	.8559
0.3	.8650	.8735	.8813	.8887	.8955	.9018	.9093	.9133	.9185	.9234
0.4	.9279	.9322	.9361	.9399	.9434	.9467	.9497	.9526	.9555	.9579
0.5	.9603	.9626	.9647	.9667	.9686	.9703	.9720	.9736	.9751	.9764
0.6	.9777	.9790	.9801	.9812	.9823	.9832	.9842	.9850	.9858	.9866
0.7	.9875	.9880	.9887	.9893	.9899	.9904	.9910	.9914	.9919	.9923
0.8	.9928	.9931	.9935	.9939	.9942	.9945	.9948	.9951	.9953	.9956
0.9	.9958	.9960	.9962	.9965	.9966	.9968	.9970	.9971	.9973	.9974
ω^2	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	∞
Φ	.9976	.9985	.9991	.9995	.9997	.9998	.9999	.9999	1.0000	1

Gakugei Faculty, Tokushima University.