# Notes on General Analysis (I)

By

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In these notes we shall first give another proofs of the radius of analyticity of the power series[*] which term by term differentiated and the Taylor expansion of the power series in the sphere of analyticity,[1] and then investigate in detail the state of the singular point of the power series on the boundary of the sphere of analyticity.[2] In the end of these papers, we shall extend the theorem of Osgood of two complex variables to the case of functions whose domains lie in product spaces of two complex Banach spaces using the classical methods.

## § 1. Radius of analyticity of the power series,

Let $E_1$, $E_2$ and $E_3$ be complex-Banach-spaces and an $E_2$ valued function $h_n(x)$ defined on $E_1$ be a homogeneous polynomial of degree n. Then the radius of analyticity $\tau$ of the power series $\sum_{n=0}^{\infty} h_n(x)$ is given by

$$\frac{1}{\tau} = \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|}\ ^{2)}.$$

We shall use following lemma for our purpose.

**Lemma.** *Suppose that $x$ and $y$ are arbitrary points respectively on $\|x\|<\tau$ and on $\|y\|=1$. Let $\rho$ be an arbitrary positive number such that $\rho<\tau-\|x\|$. Then there exists a positive number $\sigma$ which is less than 1 and satisfies the following inequalities*

$$\| h_n(x+\alpha y) \| < \sigma^n$$

*for $|\alpha| \leq \rho$ and $n \geqq n_0(\rho, \sigma)$.*[2]

Put $h_n(x+\alpha y)=\sum_{i=0}^{n} h_{n-i,i}(x,y)\,\alpha^i$. Then $h_{n-i\cdot i}(x,y)$ is a homogeneous polynomial of

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*) This is called "The radius of absolute convergence of the power series" by E. Hille ; Functional analysis and semigroups, 1948.

1) See, A. E. Taylor, (1) Analytic functions in general analysis, Annali della R. Scuola Normale Superiore di Pisa, Seri. 11 Vol. vl (1937). (2) Additions to the theory of polynomials in normed linear spaces (Tohoku M. J. 44, 1938). (3) On the properties of analytic functions in abstract spaces, Math. Ann. 115, 1938.

2) I. Shimoda : (1) On power series In abstract spaces. Mathematica Japonicae, Vol. 1, No. 2. The principal part of the proof of Theorem 2 is "Lemma" in this paper. (2) On the behaviour of power series on the boundary of the sphere of analyticity in abstract spaces, Proceeding of Japan A. Vol. 27 (1951), No. 2. or, Journal of Science of Gakugei Faculty, Tokusima University, Vol. 1, 1950.

degree n-i with respect to x and a homogeneous polynomial of degree i with respect to y. $h_{n-1,1}(x, y)$ is the differential of $h_n(x)$ with increment y.

**Theorem 1.** *The radius of analyticity of* $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ *with respect to* $x$ *and independent of* $y$ *is* $\tau$.

*Proof.* The radius of analyticity $\tau'$ of $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ independent of y is clearly

$$\frac{1}{\tau'} = \sup_{y} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n-1]{\|h_{n-1,1}(x, y)\|}.$$

Now put $y = \|y\| \cdot y'$, then

$$\frac{1}{\tau'} = \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n-1]{\|h_{n-1,1}(x, y')\| \cdot \|y\|}$$

$$= \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, y')\|}.$$

When $x = y'$, $h_{n-1,1}(x, x) = nh_n(x)$.[3] Therefore

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} = \overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{n}} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, x)\|}$$

$$\leq \sup_{\|y'\|=1} \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_{n-1,1}(x, y')\|}$$

and we have $\frac{1}{\tau} \leq \frac{1}{\tau'}$. That is, $\tau \geqq \tau'$.

Let x and y be arbitrary points respectively in $\|x\| < \tau$ and $E_1$. Since there exists a positive number $\rho'$ such that $0 < \rho' \|y\| < \tau - \|x\|$, we have

$$\| h_n(x + \alpha y) \| < \sigma^n,$$

for $|\alpha| \leq \rho'$ and $n \geqq n_0$, by lemma, where $0 < \sigma < 1$. Thus we have, $\| h_{n-1,1}(x, y) \| < \frac{1}{\rho'} \sigma^n$, for $n \geqq n_0$. This shows that $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ is absolutely convergent in $\|x\| < \tau$ for an arbitrary fixed y and we see that $\sum_{n=1}^{\infty} h_{n-1,1}(x, y)$ is analytic in $\|x\| < \tau$. That is, $\tau \leq \tau'$. Then we have $\tau = \tau'$.

Corollary. Let $h_{n-i,i}(x, y_1, y_2, \ldots, y_i)$ be the i-th derivative of $h_n(x)$ with increments $y_1, y_2, \ldots y_i$. Then the radius of analyticity of $\sum_{n=i}^{\infty} h_{n-i,i}(x, y_1, y_2, \ldots, y_i)$ with respect to x and independent of $y_1, y_2, \ldots, y_i$ is $\tau$.

**Theorem 2.** *Let* $x$ *be an arbitrary point in* $\|x\| < \tau$. *Then the radius of analyticity of the Taylor expansion of* $\sum_{n=0}^{\infty} h_n(x)$ *at* $x$ *is greater than or equal to* $\tau - \|x\|$.

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3) A. E. Taylor (2), Theorems 2. 3, 2. 5 and 2. 7.

*Proof.* Let y be an arbitrary point on $\|y\|=1$, and $\rho$ be an arbitrary positive number such that $\rho < \tau - \|x\|$.

Appealing to Lemma, we have

$$\| h_n (x+\alpha y) \| < \sigma^n,$$

for $|\alpha| \leqq \rho$ and $n \geqslant n_0$, where $0 < \sigma < 1$. Then we have,

$$\| h_{n-i, i} (x, y) \| \leqq \frac{1}{\rho^i} \sigma^n \cdots\cdots (1)$$

for $\|y\|=1$ and $n \geqslant n_0$. Now, put $U_m(y) = \sum_{n=i}^{m} h_{n-i, i}(x, y)$ with $m = i, i+1, \ldots\ldots$ and for an arbitrary y in complex Banach spaces. $U_m(y)$ is a homogeneous polynomial of degree i satisfying the following inequalities

$$\begin{aligned} \| U_p(y) - U_q(y) \| &= \| \sum_{n=q+1}^{p} h_{n-i, i}(x, y) \| \\ &= \|y\|^i \cdot \| \sum_{n=q+1}^{p} h_{n-i, i}(x, y') \|, \text{ where } y' = \frac{y}{\|y\|}, \\ &\leqq \|y\|^i \cdot \sum_{n=q+1}^{p} \frac{1}{\rho^i} \sigma^n, \text{ from } (1), \\ &\leqq \left(\frac{\|y\|}{\rho}\right)^i \frac{1}{1-\sigma} \sigma^{q+1}, \end{aligned}$$

for $p > q \geqslant n_0$. This shows that the sequence $\{U_m(y)\}$ is convergent on whole spaces, and we see that $\lim_{m\to\infty} U_m(y) = \sum_{n=i}^{\infty} h_{n-i, i}(x, y)$ is a homogeneous polynomial of degree i with respect to y.[4] Put $h'_i(y) = \sum_{n=i}^{\infty} h_{n-i,i}(x, y)$ and let $\tau'$ be the radius of analyticity of the power series $\sum_{i=0}^{\infty} h'_i(y)$, then we have

$$\begin{aligned} \frac{1}{\tau'} &= \sup_{\|y\|=1} \overline{\lim_{i\to\infty}} \sqrt[i]{\| h'_i(y) \|} \\ &\leqq \sup_{\|y\|=1} \overline{\lim_{i\to\infty}} \sqrt[i]{\frac{1}{\rho^i} \frac{\sigma^i}{1-\sigma}}, \text{ from } (1), \\ &\leqq \frac{\sigma}{\rho} < \frac{1}{\rho}. \end{aligned}$$

Thus we have $\tau' \geqslant \rho$. Since $\tau - \|x\| - \rho$ can be taken as small as we like, we have $\tau' \geqslant \tau - \|x\|$. This completes the proof.

## § 2. Singular point of power series**)

The radius of analyticity $\tau$ of the power series $\sum_{n=0}^{\infty} h_n(x)$ is given by following equation[5]

$$\frac{1}{\tau} = \sup_{G \in K} \overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x \in G} \| h_n(x) \|},$$

4) A. E. Taylor (2), Theorem 3. 7.

5) See : (2) of 2).

where G is an arbitrary compact set extracted from the set $\|x\|=1$ and K is composed of all such compact sets. The sphere $\|x\| < \tau$ is called the sphere of analyticity of $\sum_{n=0}^{\infty} h_n(x)$.

**Theorem 3.** *Suppose that a compact set G exists on the boundary of the sphere of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$, which satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \| h_n(x) \|} = 1 . \quad \cdots\cdots(2)$$

*Then we can find a sequence $\{x_n\}$, which converges to $x_0$ and satisfies the equation $\overline{\lim_{n_i\to\infty}} \sqrt[n_i]{\|h_{n_i}(x_i)\|} = 1$, in G and at least a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on the set M composed of $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.*

*Proof.* From the assumption (2), we have

$$\sqrt[n_i]{\sup_{x\in G} \| h_{n_i}(x) \|} > \frac{1}{1+\varepsilon_i}, \quad \cdots\cdots(3)$$

for a sequence of positive number $\varepsilon_i$, which tends to zero, where $n_i$ depends on $\varepsilon_i$ for $i=1, 2, \ldots\ldots$

Since G is compact and $h_{n_i}(x)$ is continuous on C, there exists $x_i$ in G which satisfies

$$\| h_{n_i}(x_i) \| = \sup_{x\in G} \| h_{n_i}(x) \| .$$

Since $\{x_i\}$ is a subset of G, we can select a subsequence of $\{x_i\}$ which converges in G. In order not to change notation, we shall suppose simply that the sequence $\{x_i\}$ itself converges to $x_0$, which is the element of G. Then, from the construction of $\{n_i\}$, we have $\lim_{n_i\to\infty} \sqrt[n_i]{\| h_{n_i}(x_i)\|} = 1$, and $x_i \longrightarrow x_0$.

Put $x_i(1+\varepsilon_i)=y_i$, then $y_i$ converges to $x_0$. From (3), we have

$$\| h_{n_i}(y_i) \| > 1 \quad \cdots\cdots(4)$$

If $\sum_{n=0}^{\infty} h_n(x)$ has not a singular point on M, which composed of $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$, $\sum_{n=0}^{\infty} h_n(x)$ is analytic on M. Therefore, for an arbitrary positive number $\varepsilon$ and $\theta$ $(0 \leqq \theta \leqq 2\pi)$, there exists $N_\theta$, such that

$$\| \sum_{n=0}^{\infty} h_n(x) \| \leqq N_\theta ,$$

for $\| x-x_0 e^{i\theta} \| < \varepsilon$. By the covering theorem of Heine-Borel for a compact set, we can find finite points $x_0 e^{i\theta_1}$, $x_0 e^{i\theta_2}, \ldots\ldots, x_0 e^{i\theta_m}$, such that $\| \sum_{n=0}^{\infty} h_n(x) \| \leqq N$, for $\|x-x_0 e^{i\theta_j} \| \leqq \varepsilon$, where $j=1, 2, 3, \ldots\ldots, m$ and $N=\max(N_{\theta_1}, N_{\theta_2}, \ldots\ldots, N_{\theta_m})$. Now we choose two positive numbers $\rho$ and $\delta$, such that $\|\alpha x-x_0 e^{i\theta_j} \| < \varepsilon$, when $\| x-x_0 \| < \rho (< \varepsilon)$, $|\alpha| = 1+\delta$

and suitable $\theta_j$ is chosen from $\theta_1, \theta, \ldots, \theta_m$ for $\alpha$.

Then we have

$$\| h_n(x) \| = \| \frac{1}{2\pi i} \int_{|\alpha|=1+\delta} \frac{\sum_{n=0}^{\infty} h_n(x)}{\alpha^{n+1}} d\alpha \| \leqq \frac{N}{(1+\delta)^n} \cdots\cdots\cdots\cdots(5)$$

for $\|X-X_0\| < \rho$ and $n=1, 2, \ldots\ldots$

Since $y_i$ converges to $X_0$, (5) contradicts to (4). This shows that $\sum_{n=0}^{\infty} h_n(x)$ has at least a singular point on M. Here $x$ is not necessarily a singular point of $\sum_{n=0}^{\infty} h_n(x)$, as a following example shows. Put $h_n(X)=x^{n-1}y$ in the complex-2-dimensional spaces, then $h_n(X)$ is a homogeneous polynomial of degree n, where $X=(x, y)$. Then the radius of analyticity of $\sum_{n=1}^{\infty} h_n(x)$ is 1. Let G be a compact set on $\|X\|=1$ composed of $X_0=(e^{i\theta}, 0)$ and $X_m=\left(\sqrt{1-\frac{1}{m}}e^{i\theta}, \sqrt{\frac{1}{m}}e^{i\theta}\right)$ with $m=1, 2, 3, \ldots\ldots$

$$\sup_{X\in G} \| h_n(X) \| = \sup_{m} \left| \left(1-\frac{1}{m}\right)^{\frac{n-1}{2}} e^{i(n-1)\theta} \left(\frac{1}{m}\right)^{\frac{1}{2}} e^{i\theta} \right|$$

$$= \sup_{m} \left(1-\frac{1}{m}\right)^{\frac{n-1}{2}} \left(\frac{1}{m}\right)^{\frac{1}{2}}$$

$$= \left(1-\frac{1}{n}\right)^{\frac{n-1}{2}} \left(\frac{1}{n}\right)^{\frac{1}{2}}$$

because $(1-t)^{n-1}t$ takes its maximum at $t=\frac{1}{n}$ in the interval $0 \leqq t \leqq 1$. Since $\left(1-\frac{1}{n}\right)^{\frac{n-1}{2}}\left(\frac{1}{n}\right)^{\frac{1}{2}} = \| h_n(X_n) \|$, we have

$$\sup_{X\in G} \| h_n(X) \| = \| h_n(X_n) \| .$$

On the other hand, $\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X_n) \|} = \overline{\lim_{n\to\infty}} \left(1-\frac{1}{n}\right)^{\frac{n-1}{2n}} \left(\frac{1}{n}\right)^{\frac{1}{2n}} = 1$, and moreover $X_m$ converges to $X_0$. Nevertheless, it is not $X_0$ but $X_0 e^{-i\theta}$, that is a singular point of $\sum_{n=1}^{\infty} h_n(X)$.

**Corollary.** *Suppose that a compact set $G$ exists on $\|x\|=1$ such that*

$$\frac{1}{\tau} = \overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{X\in G} \| h_n(X) \|} ,$$

*then we can find a sequence $\{x_n\}$, which converges to $x_0$ in G and satisfies $\overline{\lim_{n\to\infty}} \sqrt[n_i]{\| h_{n_i}(X_i) \|} = \frac{1}{\tau}$, and at least a singular point on the set composed of $x_0 \tau e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.*

**Theorem 4.** *If a point $x$, which lies on the boundary of the sphere of analiticity of $\sum_{n=0}^{\infty} h_n(X)$, satisfies the following equality $\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X) \|} = 1$,* *then there exists at least a singular point on the circle* $x e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

*Proof.* Since $x$ is a compact set, $\sup \| h_k(X) \| = \| h_k(X) \|$ and $\{x\}$ converges to $x$. Therefore, Theorem 3 is applicable and we see that Theorem 4 is true.

**Corollary,** *If a point* $x$, *which lies on* $\| x \| = 1$, *satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\| h_n(X) \|} = \frac{1}{\tau},$$

*then there exists at least a singular point on the circle* $x \tau e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

As well as the case of Theorem 3, $x\tau$ is not necessarily a singular point, as we can easily find an opposite example in the power series of complex numbers.

## § 3. Analytic functions of two variables

**Lemma.** *Let* $\{f_n(x)\}$ *be a sequence of functions on* $E_1$ *to* $E_2$, *each being analytic in a domain D, and convergent to* $f(X)$ *in D. If on each compact set G extracted from D the members of the sequence possess a commom bound M,* $f(x)$ *is analytic in D.*

*Proof.* Let $x_0$ be any point in D, then there exists a pair of positive numbers $\rho$, $M$, for which $\| f_n(x) \| \leqq M$ $(n=1, 2, \ldots)$, when $\|x-x_0\| \leqq \rho$ in D. If not so, there exists a subsequence $\{f_m(x)\}$ of $\{f_n(x)\}$ and a sequence $\{x_m\}$, which tends to $x_0$, such that $\| f_m(x_m) \| \geqq m$. On the other hand, since $\{x_m\}$ is a compact set, $f_n(x)$ must be bounded on $\{x_m\}$ in contradiction to $\| f_m(x_m) \| \geqq m$. Then $f(x)$ is analytic in $\| x-x_0 \| < \rho$ by the thorem of A. E. Taylor.[6] Therefore $f(x)$ is analytic in D.

**Theorem 5.** *A function* $f(x, y)$ *defined in a domain D of* $E_1 \times E_2$ *with values of* $E_3$ *is analytic in D if the following conditions are satisfied,* 1) $f(x,y)$ *is analytic with respect to* $x, y$ *separately in D,* 2) *let G be any compact set extracted from D, then there exists a positive number* $M_G$ *such that* $\| f(x, y) \| \leqq M_G$ *on G.*

*Proof.* Let $(x_0, y_0)$ be any point of D. We can choose two positive numbers R, S such that a domain $\| x-x_0 \| < R$, $\| y-y_0 \| < S$ is contained in D. Then it suffices to show

6) A. E. Taylor (3), loc. cit. page 469. Theorem 15. Let $\{f_n(x)\}$ be a sequence of functions on $E_1$ to $E_2$, each analytic in a domain D of $E_1$, and convergent to a limit f (x) in D. If in each region interior to D the members of the sequence possess a common bound, f (x) is analytic in D.

that $f(x, y)$ is analytic in a domain $\| x-x_0 \| < R$, $\| y-y_0 \| < S$. Without losing generality, we may assume that $(x_0, y_0)=(0, 0)$. If $x$ is an arbitrary fixed point of $\|x\| < R$, $f(x, y)$ is an analytic function of $y$ in $\|y\| < S$. Therefore we have

$$f(x, y) = \sum_{n=0}^{\infty} U_n(x, y),$$

where $U_n(x, y)$ is a homogeneous polynomial of degree $n$ with respect to $y$. Obviously $U_0(x, y)=f(x, 0)$, which is analytic with respect to $x$. If $y$ is an arbitrary fixed point of $\|y\| < S$, there exists a positive number $\rho$ such that $\rho \|y\| < S$. Then we have

$$U_1(x, y)=\frac{1}{2\pi i}\int \frac{f(x, \alpha y)}{\alpha^2} d\alpha,$$

the integral being taken in the positive sense on the circle $|\alpha|=\rho$. Now we define $S_m(x)=\frac{1}{2\pi i}\left\{\sum_{i=1}^{m}\frac{f(x, \eta_i y)}{\eta_i^2}(\xi_{i+1}-\xi_i)\right\}$ for $m=1, 2, \ldots,$ where $\xi_1, \xi_2, \ldots, \xi_m, \xi_{m+1}(=\xi_1)$ lie on the circle $|\alpha|=\rho$, and each $\eta_i$ lies on the arc $\widehat{\xi_i \xi_{i+1}}$, and $\max_{1\leq i\leq m} |\xi_{i+1}-\xi_i|$ tends to zero when $m$ tends to infinity. Then

(1), $S_m(x)$ is analytic in $\|x\| < R$,

(2). if $x$ is an arbitrary fixed point of $\|x\| < R$, $\lim_{m\to\infty} S_m(x)=U_1(x, y)$,

(3). let $G_0$ be any compact set extracted from the sphere $\|x\| < R$, and $T$ be a set of $\alpha y$, where $|\alpha|=\rho$, $G=(G_0, T)$ is a compact set in D. By the hypothesis 2) there exists a positive number $M_G$ such that $\| f(x, \alpha y) \| \leqq M_G$, for $(x, \alpha y)$ on G. Therefore

$$\| S_m(x) \| \leqq \frac{M_G}{\rho}.$$

Thus the lemmer is applicable, and we see that $U_1(x, y)$ is analytic with respect to $x$. On the other hand, $U_1(x, y)$ is linear with respect to $y$, and we see by the theorem of Kerner[7] that $U_1(x, y)$ is continuous in $(\|x\| < R, E_1')$. Generally $U_n(x, y)=\frac{1}{n!}\left[\delta^n f(x; y_1, y_2, \ldots, y_n)\right]$, where

$$\delta^n f(x; y_1, y_2, \ldots, y_n)=\frac{1}{(2\pi i)^n}\int\frac{d\alpha_1}{\alpha_1^2}\int\frac{d\alpha_2}{\alpha_2^2}\cdots\cdots\int\frac{f(x, \alpha_1 y_1+\cdots\cdots+\alpha_n y_n)}{\alpha_n^2} d\alpha_n,$$

each integral being taken in the positive sense on the circle $|\alpha_i|=\rho'$ for $i=1, 2, \ldots, n$, where $\rho'$ must satisfy $\|\alpha_1 y_1+\cdots\cdots+\alpha_n y_n\| < S$ when $|\alpha_i| \leqq \rho'$ $(i=1, 2, \ldots, n)$. Repeating the process described in the proof of the continuity of $U_1(x, y)$, we see that $\delta^n f(x; y_1, y_2, \ldots, y_n)$ is continuous with respect to $(x, y_1, y_2, \ldots, y_n)$. Now let G be any compact set extracted from $(\|x\| < R, \|y\| < S)$, and $G_0$, $G_1$ be the projections of G into $\|x\| < R$ and $\|y\| < S$ respectively. Since $G_0$, $G_1$ are clearly the compact sets in $\|x\| < R$, $\|y\| < S$ respectively, it follows that $\max_{G\ni y} \|y\|=s < S$. Let C be a circle $|\alpha|=\rho$ (where $1 < \rho < \frac{S}{s}$), $C\times G_1$ is a

7) M. Kerner, Zur Theorie der impliziten funktional Operation, Studia Math. T. III. (1931)

compact set in $\|y\| < S$ and $G' = (G_0, C \times G_1)$ is a compact set in $(\|x\| < R, \|y\| < S)$. Then there exists a positive number $M_{G'}$ such that $\| f(x, y) \| < M_{G'}$ when $(x, y) \in G'$. Obviously $G$ is contained in $G'$, and this shows

$$\| U_n(x, y) \| = \| \frac{1}{2\pi i} \int_c \frac{f(x, \alpha y)}{\alpha^{n+1}} d\alpha \| \leqq \frac{M_{G'}}{\rho^n}$$

for $n = 1, 2, 3, \ldots,$ when $(x, y) \in G$. Thus the function $f(x, y) = \sum_{n=0}^{\infty} U_n(x, y)$ converges uniformly on $G$, and so $f(x, y)$ is continuous in $(\|x\| < R, \|y\| < S)$. This completes the proof.

**Corollary.** *If $E_2$-valued function $f(x, y)$ is analytic with respect to $x, y$ separately and bounded in the domain $D$ of $E_1 \times E_2$, $f(x, y)$ is analytic in $D$.*

**Remark.** By using Theorem 5 and the theorem of B-continuity of Zorn,[8] *the generalized Hartogs's theorem* can be proved as in the classical methods. Let $f(x, y)$ is analytic with respect to each variables separately, then there exists an open set $V$, in which $f(x, y)$ is bounded, in an arbitrary neighbourhood $U$ of any point $(x, y)$ in the domain. Appealing to Theorem 5, $f(x, y)$ is analytic in $V$. Therefore $f(x, y)$ is B-continuous and then $f(x, y)$ is analytic with respect to $(x, y)$ by the Theorem of Zorn, because $f(x, y)$ is G-differentiable.

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**) A power series $\sum_{n=0}^{\infty} h_n(x)$ is called analytic at a point $x$ when there exists at least a neighbourhood $V(x)$ of $x$, on which $\sum_{n=0}^{\infty} h_n(x)$ is continuous strongly and G-differentiable. A point $x$ is called a singular point of $\sum_{n=0}^{\infty} h_n(x)$, when $\sum_{n=0}^{\infty} h_n(x)$ is not analytic at a point $x$.

8) Max A. Zorn : Characterization of Analytic Functions in Banach Spaces, Annals of Math. (2) 46 (1945). In the paper, the generalized Hartogs's theorem was proved very elegantly by Zorn.