On the System of Semigroup Operations Defined in a Set.

By

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§ 1. Introdution.

The object of this paper is the semigroup operation system $\mathfrak{M}$ of a set E, i. e., the aggregate of all possible semigroup operations λ, μ, $\cdots$ defined in abstract set E. More strictly,

Definition 1. $\mathfrak{M}$ is the set of all λ satisfying the below conditions :

(1) To each pair of elements a and $b \in E$ corresponds a unique element $a \lambda b \in E$.

(2) λ is associative : $(a\lambda b)\lambda c = a\lambda(b\lambda c)$ for any $a, b, c \in E$.

The equality of elements of $\mathfrak{M}$ is defined as follows.

Definition 2. Two cperations λ and μ are said to de equal i. e $\lambda=\mu$, if $x\lambda y = x\mu y$ for any $x, y \in E$.

In the present paper we shall discuss how the semigroup opertaion system is ordered, and how we realize the ordering in the transformation semigroups, but there remain many problems unsolved. In order to intrcduce some quasi-ordering into the system we will restrict ourselves to the universal semigroup operation system $\mathfrak{U}$ of E.

Definition 3. A semigroup operation λ defined in E is called univesal if for any $c \in E$ there exist a and $b \in E$ such that $a \lambda b = c$.

By the universal semigroup operation system $\mathfrak{U}$ of E is meant the set of all universal semigroup operations defined in E.

§ 2. The Necessary and Sufficient Condition of a Semigroup.

As the preliminaries we shall relate the necessary and sufficient condition [1] that the associative law is fulfilled by an algebraic system $E(\lambda)$[(1)] by which is meant an abstract set with a binary operation λ. If F is a subset of the algebraic system E and $a \lambda b \in F$ whenever a and $b \in F$, we call F an algebraic subsystem of E. Although it is needless to say,

Lemma 1. An algebraic subsystem F of a semigroup E is a subsemigroup of E.

Lemma 2. If a semigroup E is homomorphic or anti-homomorphic on an algebraic system E', then E' is a semigroup.

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1) we denote by $E(\lambda)$ the algebraic system E with λ when λ need to be specially assigned, but simply by E when there is no fear of confusion.

Definition 4. A single-valued mapping T of an abstract set M into itself is called a transformation of M, i. e. , to any $x \in M$ corresponds a unique element $Tx \in M$. Of course we define equality of two transformations as

$$T = S \quad \text{if} \quad Tx = Sx \quad \text{for all} \quad x \in M.$$

If the product $R = TS$ of transformations T and S is given as $Rx = S(Tx)$ for $x \in M$, then the set of all transformations of M obviously forms a semigroup [2], whence the set is called the transformation semigroup $\mathfrak{T}_M$ on M, and a subsemigroup of $\mathfrak{T}_M$ is called a transformation subsemigroup on M.

As the special transformation system, we define a realization system and a faithful realization system as following.

Definition 5. Let a be an element of the algebraic system E with an operation λ. Then the transformation $R_\lambda(a)$ given as $R_\lambda(a)x = x\,\lambda\,a$ (for $x \in E$) is called the right λ—realization of a in E, the transformation $L_\lambda(a)$ given as $L_\lambda(a)\,x = a\,\lambda\,x$ (for $x \in E$) is called the left λ—realization of a in E.

Letting $\mathfrak{R}_\lambda = [R_\lambda(a) \mid a \in E]$, $\mathfrak{L}_\lambda = [L_\lambda(a) \mid a \in E]$,

$\mathfrak{R}_\lambda$ (or $\mathfrak{L}_\lambda$) is called the right (left) λ—realization system of $E(\lambda)$, affording little convenience to our general discussion [3].

Definition 6. Let $\bar{E}(\bar{\lambda})$ be the extended algebraic system of $E(\lambda)$, which is obtained by abjoining only one new element p to $E(\lambda)$ and defining the operation $\bar{\lambda}$ in $\bar{E}$ as follows.

$$x\,\bar{\lambda}\,y = x\,\lambda\,y \qquad \text{if} \quad x, y \in E,$$
$$p\,\bar{\lambda}\,x = x\,\bar{\lambda}\,p = x \qquad \text{if} \quad x \in \bar{E}.$$

As easily shown, $\bar{E}$ is a semigroup if and only if E is a semigroup [4].

Definition 7. Let $a \in E(\lambda) \subset \bar{E}(\bar{\lambda})$. The right (left) λ— realization of a in $\bar{E}(\bar{\lambda})$ is called right (left) faithful λ—realization of a, written $\bar{R}_\lambda(a)$ ($\bar{L}_\lambda(a)$); and the set of them i. e., $\bar{\mathfrak{R}}_\lambda = [\bar{R}_\lambda(a) \mid a \in E]$ or $\bar{\mathfrak{L}}_\lambda = [\bar{L}_\lambda(a) \mid a \in E]$ is called the right or left faithful λ—realization system of E respectively, where $a \leftrightarrow \bar{R}_\lambda(a)$ or $a \leftrightarrow \bar{L}_\lambda(a)$ is one-to-one.

Now we have the following theorems.

Theorem 1. *An algebraic system* $E(\lambda)$ *is a semigroup if and only if* $\bar{R}_\lambda(a)\,\bar{R}_\lambda(b) = \bar{R}_\lambda(a\,\lambda\,b)$ *for every* $a, b \in E$.

Theorem 1'. *An algebraic system* $E(\lambda)$ *is a semigroup if and only if* $\bar{L}_\lambda(a)\,\bar{L}_\lambda(b) = \bar{L}_\lambda(b\lambda a)$ *for every* $a, b \in E$.

Remark. The formula shows that $\bar{\mathfrak{R}}_\lambda$ ($\bar{\mathfrak{L}}_\lambda$) is an algebraic system and $E(\lambda)$ is isomorphic (anti-isomorphic) on $\bar{\mathfrak{R}}_\lambda$ ($\bar{\mathfrak{L}}_\lambda$).

Proof of Theorem 1. Suppose that $E(\lambda)$ is a semigroup. By the assumption of λ and the definition of $\bar{R}_\lambda$,

$$\left\{\bar{R}_\lambda(a)\bar{R}_\lambda(b)\right\}x=\bar{R}_\lambda(b)\left\{\bar{R}_\lambda(a)x\right\}=\bar{R}_\lambda(b)(x\lambda a)=(x\lambda a)\lambda b$$

$$=x\lambda(a\lambda b)=\bar{R}_\lambda(a\lambda b)x \qquad \text{for} \quad x\in E,$$

and $\left\{\bar{R}_\lambda(a)\bar{R}_\lambda(b)\right\}p=\bar{R}_\lambda(b)\left\{\bar{R}_\lambda(a)p\right\}=\bar{R}_\lambda(b)a=a\lambda b=\bar{R}_\lambda(a\lambda b)p.$

In short, $\left\{\bar{R}_\lambda(a)\bar{R}_\lambda(b)\right\}x=\bar{R}_\lambda(a\lambda b)x$ for any $x\in\bar{E}$.

Finally we have $\qquad \bar{R}_\lambda(a)\bar{R}_\lambda(b)=\bar{R}_\lambda(a\lambda b). \qquad (1)$

Conversely suppose (1). It follows from (1) that $\bar{\mathfrak{R}}_\lambda$ is an algebraic subsystem of the semigroup $\mathfrak{T}_{\bar{E}}$ and thet $\bar{\mathfrak{R}}_\lambda$ is isomorphic on $E(\lambda)$ under the mapping: $\bar{R}_\lambda(x)\leftrightarrow x$. Hence $E(\lambda)$ is immediately concluded to be a semighoup by means of Lemma 1 and 2.

We can similarly prove Theorem 1′. If the correspondence between $E(\lambda)$ and its realization system $\mathfrak{R}_\lambda(\mathfrak{L}_\lambda)$ is one-to-one , $\mathfrak{R}_\lambda(\mathfrak{L}_\lambda)$ is isomorphic with the faithful realization system $\bar{\mathfrak{R}}_\lambda(\bar{\mathfrak{L}}_\lambda)$. Therefore we have.

Corollary 1. *Assum that* $a\leftrightarrow R_\lambda(a)$ *is one-to-one. In order that* $E(\lambda)$ *is a semigroup, it is necessary and sufficient that* $R_\lambda(a)R_\lambda(b)=R_\lambda(a\lambda b)$ *for every* $a, b\in E$.

Corollary 2. *Assume that* $a\leftrightarrow L_\lambda(a)$ *is one-to-one. In ordey that* $E(\lambda)$ *is a semigroup, it is necessary and sufficient that* $L_\lambda(a)L_\lambda(b)=L_\lambda(b\lambda a)$ *for every* $a, b\in E$.

§ 3. The Ordering in $\mathfrak{U}$.

Definition 8. If $(a\lambda b)\mu c=a\lambda(b\mu c)$ for any $a, b, c\in E$, then we denote it by $\lambda\gtrsim\mu$, or by $\mu\lesssim\lambda$.

Theorem 2. *Let* $\lambda, \mu\in\mathfrak{M}$ *and* $\mu\in\mathfrak{U}$. *If* $\lambda\gtrsim\mu$ *and* $\mu\gtrsim\nu$, *then* $\lambda\gtrsim\nu$.

Proof. For any a, b and $c\in E$,

$$\begin{aligned}(a\lambda b)\nu c&=\left\{a\lambda(b'\mu b'')\right\}\nu c && (\because\ \mu\in\mathfrak{U},\ b=b'\mu b'')\\ &=\left\{(a\lambda b')\mu b''\right\}\nu c && (\because\ \lambda\gtrsim\mu)\\ &=(a\lambda b')\mu(b''\nu c) && (\because\ \mu\gtrsim\nu)\\ &=a\lambda\left\{b'\mu(b''\nu c)\right\} && (\because\ \lambda\gtrsim\mu)\\ &=a\lambda\left\{(b'\mu b'')\nu c\right\} && (\because\ \mu\gtrsim\nu)\\ &=a\lambda(b\nu c).\end{aligned}$$

Moreover it always holds that $\lambda \gtrsim \lambda$ for every $\lambda \in \mathfrak{M}$. If we are confined to the universal semigroup operation system $\mathfrak{U}$, the relation $\gtrsim$ is a quasi-ordering [5] in $\mathfrak{U}$. Let us identify λ and μ, denote $\lambda \sim \mu$, when $\lambda \gtrsim \mu$ as well as $\lambda \lesssim \mu$. Then $\mathfrak{U}$ becomes a partially ordered set under the identification [6].

Remark. The universality of μ in Theorem 2 has an effect on the transitive law. More precisely, if it were not for the universality, the law would not necessarily hold [7]. Let us take for example the finite set M (of three elements a, b, and c) in which the three semigroup operations λ, μ[2] and ν are given as the below product tables show.

λ

right / left	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

μ

right / left	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

ν

right / left	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

1° At first we must show that λ, μ, and ν are all semigroup operations. In fact, respecting μ, it is evident; as far as λ, μ are concerned, we can prove them easily dy Theorem 1 or dy direct method [8].

2° $\lambda \gtrsim \mu$

For, x, y, and z symbolling one of a, b, and c,

$(x \lambda y) \mu z = a$, $x \lambda (y \mu z) = x \lambda a = a$. Hence $(x \lambda y) \mu z = x \lambda (y \mu z)$.

3° $\mu \gtrsim \nu$

For, $(x \mu y) \nu z = a \nu z = a$, $x \mu (y \nu z) = a$, Hence $(x \mu y) \nu z = x \mu (y \nu z)$.

4° On the other hand [9] $\lambda \not\gtrsim \nu$.

For, $(b \lambda c) \nu b = b \nu b = b$, $b \lambda (c \nu b) = b \lambda a = a$. Therefore $(b \lambda c) \nu b \neq b \lambda (c \nu b)$.

Now let us define α, β as following:

$$x \alpha y = y \quad , \quad x \beta y = x \qquad \text{for every } x,\ y \in E,$$

where α, β is easily shown to belong to $\mathfrak{U}$. Then we have

Corollary 3. *$\alpha \gtrsim \lambda$ and $\lambda \gtrsim \beta$ for every $\lambda \in \mathfrak{M}$.*

Proof. For any x, y, $z \in E$,

$$(x \alpha y) \lambda z = y \lambda z, \qquad x \alpha (y \lambda z) = y \lambda z,$$

and

$$(x \lambda y) \beta z = x \lambda y, \qquad x \lambda (y \beta z) = x \lambda y \quad ;$$

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2) μ is not univeisal.

hence $(x \alpha y) \lambda z = x \alpha (y \lambda z), \quad (x \lambda y) \beta z = x \lambda (y \beta z)$.

Consequently we can assert that $\mathfrak{U}$ *is the above and below bounded partially ordered set under the mentioned adequate identification.*

§ 4. The Problem of Ordering in the Realization System.

As the validity of associative law with respect to one operation has been reduced to the problem in the faithfull realization system (cf. § 2, Theorem 1, 1′), so the comparability[3] between different operations defined in E will be considered as that between different realization systems of E.

Theorem 3. In order that $\lambda \gtrsim \mu$ for $\lambda, \mu \in \mathfrak{M}$, it is necessary and sufficient that

(1) $\bar{R}_\lambda(a) \bar{R}_\mu(b) = \bar{R}_\lambda(a \mu b)$ *for every* a *and* $b \in E$.

Proof. Suppose that $\lambda \gtrsim \mu$. If $x \in E \subset \bar{E}$,

$$\left\{\bar{R}_\lambda(a) \bar{R}_\mu(b)\right\} x = \bar{R}_\mu(b) \left\{\bar{R}_\lambda(a) x\right\} = \bar{R}_\mu(b)\,(x \lambda a)$$

$$= (x \lambda a) \mu b = x \lambda (a \mu b) = \bar{R}_\lambda(a \mu b) x \;;$$

otherwise, $\left\{\bar{R}_\lambda(a) \bar{R}_\mu(b)\right\} p = \bar{R}_\mu(b) \left\{\bar{R}_\lambda(a) p\right\} = \bar{R}_\mu(b) a = a \mu b = \bar{R}_\lambda(a \mu b) p$,

after all $\left\{\bar{R}_\lambda(a) \bar{R}_\mu(b)\right\} x = \bar{R}_\lambda(a \mu b) x$ for any $x \in \bar{E}$.

Therefore we get $\bar{R}_\lambda(a) \bar{R}_\mu(b) = \bar{R}_\lambda(a \mu b)$.

Conversely if (1) holds, then we shall arrive at

$$\bar{R}_\lambda \left\{(a \lambda b) \mu c\right\} = \bar{R}_\lambda \left\{a \lambda (b \mu c)\right\}.$$

For every a, b, and $c \in E$,

$$\bar{R}_\lambda \left\{a \lambda (b \mu c)\right\} = \bar{R}_\lambda(a) \bar{R}_\lambda(b \mu c) \qquad \text{(by Tteorem 1)}$$

$$= \bar{R}_\lambda(a) \left\{\bar{R}_\lambda(b) \bar{R}_\mu(c)\right\} \qquad \text{(by (1))}$$

$$= \left\{\bar{R}_\lambda(a) \bar{R}_\lambda(b)\right\} \bar{R}_\mu(c) \qquad \text{(by the associative law in } \mathfrak{T}_{\bar{E}})$$

$$= \bar{R}_\lambda(a \lambda b) \bar{R}_\mu(c) \qquad \text{(by Theorem 1)}$$

$$= \bar{R}_\lambda \left\{(a \lambda b) \mu c\right\}. \qquad \text{(by (1))}$$

Since the correspondence $\bar{R}_\lambda(x) \to x$ is one to one, we have

$$(a \lambda b) \mu c = a \lambda (b \mu c).$$

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3) Two operations λ and μ are said to be comparable if either $\lambda \gtrsim \mu$ or $\lambda \lesssim \mu$; and are said to be incomparable if neither $\lambda \gtrsim \mu$ nor $\lambda \lesssim \mu$, denoted $\lambda \not\gtrless \mu$.

Thus the proof of this theorem has been completed.

Similarly we get

Theorem 3'. In order that $\lambda \gtrsim \mu$ for $\lambda, \mu \in \mathfrak{M}$, it is necessary and sufficient that

(1') $\bar{L}_\mu(a)\bar{L}_\lambda(b)=\bar{L}_\mu(b\,\lambda\,a)$ for every a and $b \in E$.

we note that the above theorems need no assumption of universality and that they are the extensions of Theorem 1 and 1'. In order to establish Theorem 4 and 4' equivalent to Theorem 3 and 3', a few definitions have to be prepared.

Definition 9. The one-to-one correspondence $\bar{R}_\lambda(a) \leftrightarrow \bar{R}_\mu(a)$ between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$ is called the natural correspondence between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$. The natural correspondence between $\bar{\mathfrak{L}}_\lambda$ and $\bar{\mathfrak{L}}_\mu$ is also similarly defined.

Definition 10. Let A and B be twe subsets of a set and Φ be the system composed of transformations φ of $A \cup B$ into itself such that $\varphi(A) \subset A$ and $\varphi(B) \subset B$. If besides Φ there is a one-to-one correspondence f between A and B, and if f is preserved by Φ-transformations, i. e. $A \ni a \overset{f}{\leftrightarrow} b \in B$ implies $A \ni \varphi(a) \overset{f}{\leftrightarrow} \varphi(b) \in B$, then the correspondence f is said to be invarient by Φ.

Now it follows from Theorem 3 (3') that $\bar{R}_\lambda(a)\bar{R}_\mu(b)$ (or $\bar{L}_\mu(a)\bar{L}_\lambda(b)$) is thought as the image of $\bar{R}_\lambda(a)$ ($\bar{L}_\mu(a)$) under the transformation meaning multiplication of $\bar{R}_\lambda(a)$ ($\bar{L}_\mu(a)$) by $\bar{R}_\mu(b) \in \bar{\mathfrak{R}}_\mu$ ($\bar{L}_\lambda(b) \in \bar{\mathfrak{L}}_\lambda$) in the right side.

We shall call it $\bar{\mathfrak{R}}_\mu$-transformations ($\bar{\mathfrak{L}}_\lambda$—transformations), which, of course, may also be applied to $\bar{R}_\mu(a)$ ($\bar{L}_\lambda(a)$).

Theorem 4. *Let $\lambda, \mu \in \mathfrak{M}$. In order that $\lambda \gtrsim \mu$, it is necessary and sufficient that*

(1) $\bar{\mathfrak{R}}_\lambda \bar{\mathfrak{R}}_\mu \subset \bar{\mathfrak{R}}_\lambda$,

(2) *the natural correspondence between $\bar{\mathfrak{R}}_\lambda$ and $\bar{\mathfrak{R}}_\mu$ is invariant by $\bar{\mathfrak{R}}_\mu$-transformations.*

Proof. Let us suppose (1) and (2). It follows from (2) that the natural correspondence $\bar{R}_\lambda(a) \leftrightarrow \bar{R}_\mu(a)$ implies $\bar{R}_\lambda(a)\bar{R}_\mu(b) \leftrightarrow \bar{R}_\mu(a)\bar{R}_\mu(b)$ for every a and $b \in E$. On the other hand, there is an element $c \in E$ such that $\bar{R}_\lambda(a)\bar{R}_\mu(b)=\bar{R}_\lambda(c)$ by (1); and Theorem 1 shows $\bar{R}_\mu(a)\bar{R}_\mu(b)=\bar{R}_\mu(a\,\mu\,b)$. Hence we have $\bar{R}_\lambda(c) \leftrightarrow \bar{R}_\mu(a\,\mu\,b)$ concluding $c=a\,\mu\,b$ due to the definition of the natural correspondence. Thus we have arrived at the formula of Theorem 3. Conversely (1) and (2) follow immediately from Theorem 3.

Similarly we have

Theorem 4'. *In order that $\lambda \gtrsim \mu$ it is necessary and sufficient that*

(1') $\bar{\mathfrak{L}}_\mu \bar{\mathfrak{L}}_\lambda \subset \bar{\mathfrak{L}}_\mu$, (2′) *the natural correspondence between* $\bar{\mathfrak{L}}_\lambda$ *and* $\bar{\mathfrak{L}}_\mu$ *is invariant by* $\bar{\mathfrak{L}}_\lambda$*- transformations.*

§ 5. Translations of Operations.

By a translation on M we mean a one-to-one transformation of M onto itself. The set of all translations of M forms a group, which is called the translation group on M, and a subgroup of which is called a translation subgroup on M. Let us denote by $\mathfrak{G}$ a translation subgroup on initially given E, and individual translation by $\mathfrak{x}$, $\mathfrak{h}$, ··etc. Then corresponding to $\mathfrak{G}$ the translation subgroup $\bar{\mathfrak{G}}$ will be defined.

Definition 11. We let a transformation $\bar{\mathfrak{x}}$ of $\mathfrak{M}$ correspond to $\mathfrak{x} \in \mathfrak{G}$ as follows:

$$\lambda \xrightarrow{\bar{\mathfrak{x}}} \lambda^{\bar{\mathfrak{x}}} \qquad \text{(for any } \lambda \in \mathfrak{M})$$

where the operation $\lambda^{\bar{\mathfrak{x}}}$ is defined as

$$a\,\lambda^{\bar{\mathfrak{x}}}\,b = (a^{\mathfrak{x}}\,\lambda\,b^{\mathfrak{x}})^{\mathfrak{x}^{-1}} \quad {}^{4)} \qquad \text{for any } a,\ b \in E.$$

Lemma 3. $(a\,\lambda^{\bar{\mathfrak{x}}}\,b)\,\lambda^{\bar{\mathfrak{x}}}\,c = a\,\lambda^{\bar{\mathfrak{x}}}\,(b\,\lambda^{\bar{\mathfrak{x}}}\,c)$.

Proof.

$$(a\,\lambda^{\bar{\mathfrak{x}}}\,b)\,\lambda^{\bar{\mathfrak{x}}}\,c = \left\{(a\,\lambda^{\bar{\mathfrak{x}}}\,b)^{\mathfrak{x}}\,\lambda\,c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{(a^{\mathfrak{x}}\,\lambda\,b^{\mathfrak{x}})\,\lambda\,c^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}},$$

$$a\,\lambda^{\bar{\mathfrak{x}}}\,(b\,\lambda^{\bar{\mathfrak{x}}}\,c) = \left\{a^{\mathfrak{x}}\,\lambda\,(b\,\lambda^{\bar{\mathfrak{x}}}\,c)^{\mathfrak{x}}\right\}^{\mathfrak{x}^{-1}} = \left\{a^{\mathfrak{x}}\,\lambda\,(b^{\mathfrak{x}}\,\lambda\,c^{\mathfrak{x}})\right\}^{\mathfrak{x}^{-1}}.$$

Utilizing that λ is associative, the given formula is proved.

Lemma 4. $\bar{\mathfrak{x}}\,\bar{\mathfrak{y}} = \overline{\mathfrak{y}\,\mathfrak{x}}$.[5]

Proof.

$$a\,\lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{y}}}\,b = (a^{\mathfrak{y}}\,\lambda^{\bar{\mathfrak{x}}}\,b^{\mathfrak{y}})^{\mathfrak{y}^{-1}} = (a^{\mathfrak{y}\mathfrak{x}}\,\lambda\,b^{\mathfrak{y}\mathfrak{x}})^{\mathfrak{x}^{-1}\mathfrak{y}^{-1}} = a\,\lambda^{\overline{\mathfrak{y}\mathfrak{x}}}\,b \qquad \text{for every } a,\ b \in E.$$

Hence $\lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{y}}} = \lambda^{\overline{\mathfrak{y}\mathfrak{x}}}$ for every $\lambda \in \mathfrak{M}$.

It follows from Lemma 3 and 4 that the operation $\lambda^{\bar{\mathfrak{x}}}$ belongs to $\mathfrak{M}$ and that any λ has $\lambda^{\overline{\mathfrak{x}^{-1}}}$ as its inverse image under the transformation $\bar{\mathfrak{x}}$. Thus $\bar{\mathfrak{x}}$ for $\mathfrak{x} \in \mathfrak{G}$ has been asserted to be a translation of $\mathfrak{M}$; moreover $\bar{\mathfrak{x}}$ becomes, in f ct, a translation of $\mathfrak{U}$. It is for this reason that the following lemma shows.

Lemma 5. *If* λ *is uuiversal,* $\lambda^{\bar{\mathfrak{x}}}$ *is uuiversal.*

Proof. Given any $c \in E$, we denote $c^{\mathfrak{x}}$ by c'. Since λ is universal, there exist a' and b' such that $c' = a'\,\lambda\,b'$. Letting $a = a'^{\mathfrak{x}^{-1}}$, $b = b'^{\mathfrak{x}^{-1}}$, we get $c = a\,\lambda^{\bar{\mathfrak{x}}}\,b$; thus $\lambda^{\bar{\mathfrak{x}}}$ is universal.

If the set of all $\bar{\mathfrak{x}}$ for $\mathfrak{x} \in \mathfrak{G}$ is denoted by $\bar{\mathfrak{G}}$, we have

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4) $a^{\mathfrak{x}}$ represents the image of a under the translation $\mathfrak{x}$ of E.

5) $\lambda^{\bar{\mathfrak{x}}\bar{\mathfrak{y}}} = (\lambda^{\bar{\mathfrak{x}}})^{\bar{\mathfrak{y}}}$, $a^{\mathfrak{x}\mathfrak{y}} = (a^{\mathfrak{x}})^{\mathfrak{y}}$.

Theorem 5. $\mathfrak{G}$ *is anti-homomorphic on* $\overline{\mathfrak{G}}$. *Accordingly* $\overline{\mathfrak{G}}$ *forms a group.*

We call $\overline{\mathfrak{G}}$ a principal translation subgroup on $\mathfrak{M}$ (or $\mathfrak{U}$) to $\mathfrak{G}$. What condition does $\mathfrak{G}$ require in order that it is anti-isomorphic on $\overline{\mathfrak{G}}$? Let $\mathfrak{A}(\lambda)$ be the group of all automorphisms in an algebraic system E with λ.

Lemma 6. $\lambda = \lambda^{\bar{\mathfrak{x}}}$ *for* $\lambda \in \mathfrak{M}$ *if and only if* $\mathfrak{x} \in \mathfrak{A}(\lambda) \cap \mathfrak{G}$.

Theorem 6. $\mathfrak{G}$ *is anti-isomorphic on* $\overline{\mathfrak{G}}$ *if and only if*

$$\bigcap_{\lambda \in \mathfrak{M}} \mathfrak{A}(\lambda) \cap \mathfrak{G} = \{\mathfrak{e}\} \qquad \text{(cf. [10])}$$ [6]

Now we define a translation subgroup $\mathfrak{P}$ other than $\overline{\mathfrak{G}}$. $\mathfrak{P}$ shall be generated by the only one translation $\mathfrak{p}$ of $\mathfrak{M}$ or $\mathfrak{U}$, where $\mathfrak{p}$ maps any λ to $\lambda^{\mathfrak{p}}$ [7] given as

$$x\, \lambda^{\mathfrak{p}}\, y = y\, \lambda\, x \qquad \text{for } x,\ y \in E.$$

The subgroup generated by $\overline{\mathfrak{G}}$ and $\mathfrak{P}$ is called the fundamental translation subgroup of $\mathfrak{M}$ (or $\mathfrak{U}$) to $\mathfrak{G}$.

Letting $\underline{\mathfrak{x}} = \bar{\mathfrak{x}}\,\mathfrak{p} = \mathfrak{p}\,\bar{\mathfrak{x}}$, we immediately have

Corollary 4. $\bar{\mathfrak{x}}\,\bar{\mathfrak{y}} = \underline{\mathfrak{x}}\,\underline{\mathfrak{y}} = \overline{\mathfrak{h}\mathfrak{x}}$, $\quad \underline{\mathfrak{x}}\,\bar{\mathfrak{y}} = \bar{\mathfrak{x}}\,\underline{\mathfrak{y}} = \underline{\mathfrak{h}\mathfrak{x}}$.

On the relations between translations of operations and the initial set E.

Corollary 5. $E(\lambda)$ is isomorphic on $E(\lambda^{\bar{\mathfrak{x}}})$, and $E(\lambda)$ is anti-isomorphic on $E(\lambda^{\underline{\mathfrak{x}}})$.

§ 6. Relations between the Ordering and Translations of Operations.

In this paragraph the comparability of operations will be proved to be invariant by translations and we shall refer to the relations between the ordering and classification by fundamental translations uuder some additional condition.

Theorem 7. *If* $\lambda \gtrsim \mu$. *then* $\lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}$, $\lambda^{\underline{\mathfrak{x}}} \lesssim \mu^{\underline{\mathfrak{x}}}$ *for any* $\mathfrak{x} \in \mathfrak{G}$.

Proof. For any a, b, and $c \in E$.

$$(a\, \lambda^{\bar{\mathfrak{x}}}\, b)\, \mu^{\bar{\mathfrak{x}}}\, c = \{(a\, \lambda^{\bar{\mathfrak{x}}}\, b)^{\mathfrak{x}}\, \mu\, c^{\mathfrak{x}}\}^{\mathfrak{x}-1} = \{(a^{\mathfrak{x}}\, \lambda\, b^{\mathfrak{x}})\, \mu\, c^{\mathfrak{x}}\}^{\mathfrak{x}-1}$$

$$= \{a^{\mathfrak{x}}\, \lambda\, (b^{\mathfrak{x}}\, \mu\, c^{\mathfrak{x}})\}^{\mathfrak{x}-1} \qquad (\because\ \lambda \gtrsim \mu)$$

$$= \{a^{\mathfrak{x}}\, \lambda\, (b\, \mu^{\bar{\mathfrak{x}}}\, c)^{\mathfrak{x}}\}^{\mathfrak{x}-1} = a\, \lambda^{\bar{\mathfrak{x}}}\, (b\, \mu^{\bar{\mathfrak{x}}}\, c). \qquad \text{Hence} \quad \lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}.$$

$$(a\, \mu^{\underline{\mathfrak{x}}}\, b)\, \lambda^{\underline{\mathfrak{x}}}\, c = \{c^{\mathfrak{x}}\, \lambda\, (a\, \mu^{\underline{\mathfrak{x}}}\, b)^{\mathfrak{x}}\}^{\mathfrak{x}-1} = \{c^{\mathfrak{x}}\, \lambda\, (b^{\mathfrak{x}}\, \mu\, a^{\mathfrak{x}})\}^{\mathfrak{x}-1}$$

$$= \{(c^{\mathfrak{x}}\, \lambda\, b^{\mathfrak{x}})\, \mu\, a^{\mathfrak{x}}\}^{\mathfrak{x}-1} \qquad (\because\ \lambda \gtrsim \mu)$$

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6) $\{\mathfrak{e}\}$ is the set composed of only identity of $\mathfrak{G}$.

7) We can prove easily that $\lambda^{\mathfrak{p}} \in \mathfrak{U}$ if $\lambda \in \mathfrak{U}$.

$$=\left\{(b\,\lambda^{\mathfrak{x}}\,c)^{\mathfrak{x}}\,\mu\,a^{\mathfrak{x}}\right\}^{\mathfrak{x}-1}=a\,\mu^{\mathfrak{x}}\,(b\,\lambda^{\mathfrak{x}}\,c). \qquad \text{Hence} \quad \lambda^{\mathfrak{x}} \lesssim \mu^{\mathfrak{x}}.$$

Definition 12. If there is a suitable $\mathfrak{x} \in \mathfrak{G}$ such that $\lambda = \mu^{\bar{\mathfrak{x}}}$, then λ and μ are said to be congruent, denoted $\lambda \equiv \mu$.

Since this binary relation $\equiv$ is obviously a equivalence relation, we can classify $\mathfrak{M}$ by it. This classification is called the classification of $\mathfrak{M}$ by $\bar{\mathfrak{G}}$, written $\mathfrak{M}/\bar{\mathfrak{G}}$, whose elements are classes $\mathcal{A}$, $\mathcal{B}$, $\cdots$ composed of operations. Here we call only $\mathfrak{U}/\bar{\mathfrak{G}}$ to account, into which a quasi-ordering is introduced similarly as that in $\mathfrak{U}$.

Definittion 13. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{U}/\bar{\mathfrak{G}}$. We denote $\mathcal{A} \gtrsim \mathcal{B}$ if for any $\lambda \in \mathcal{A}$ there exists one at least $\mu \in \mathcal{B}$ such that $\lambda \gtrsim \mu$.

It is evident that the binary relation $\mathcal{A} \gtrsim \mathcal{B}$ is a quasi-ordering in $\mathfrak{U}/\bar{\mathfrak{G}}$. By Definition 12 and Theorem 7 we readily obtain :

Theorem 8. *Definition 13, the following* (1), *and* (2) *are all equivalent.*

(1) *For any* $\mu \in \mathcal{B}$ *there exists one at least* $\lambda \in \mathcal{A}$ *such that* $\lambda \gtrsim \mu$.

(2) *There exist* $\lambda \ni \mathcal{A}$, $\mu \in \mathcal{B}$ *such that* $\lambda \gtrsim \mu$.

Now we are confined to the case that $\mathfrak{G}$ is finite.[8]

Theorem 9. *If* $\lambda \equiv \mu$, *then either* $\lambda \sim \mu$ *or* $\lambda \not\gtrless \mu$.

Proof. We suppose that λ and μ are comparable, say $\lambda \gtrsim \mu$. Since there exists $\mathfrak{x} \in \mathfrak{G}$ such that $\mu = \lambda^{\bar{\mathfrak{x}}}$ by Definition 12, it holds that $\lambda \gtrsim \lambda^{\bar{\mathfrak{x}}}$ ①; while, $\mathfrak{G}$ being finite,[9] $\bar{\mathfrak{G}}$ is finite, whose order is n. Applying translations $\bar{\mathfrak{x}}, \bar{\mathfrak{x}}^2, \cdots, \bar{\mathfrak{x}}^{n-1}$ successively to both sides of ① by means of Theorem 7, we have $\lambda \gtrsim \lambda^{\bar{\mathfrak{x}}} \gtrsim \lambda^{\bar{\mathfrak{x}}^2} \gtrsim \cdots \gtrsim \lambda^{\bar{\mathfrak{x}}^n} = \lambda$, following that $\lambda \sim \mu$.

Theorem 10. *If there exist* $\lambda \in \mathcal{A}$ *and* $\mu \in \mathcal{B}$ *such that* $\lambda \gneqq \mu$,[10] *then* $\xi \not\lesssim \eta$ *for any* $\xi \in \mathcal{A}$ *and* $\eta \in \mathcal{B}$.

Proof. Suppose that there exist $\xi \in \mathcal{A}$ and $\eta \ni \mathcal{B}$ such that $\xi \lesssim \eta$. Since $\mu = \eta^{\bar{\mathfrak{x}}}$ for a suitable $\mathfrak{x} \in \mathfrak{G}$, we have $\lambda \gtrsim \mu \gtrsim \xi^{\bar{\mathfrak{x}}}$. Theorem 9 shows that $\lambda \sim \xi^{\bar{\mathfrak{x}}}$, and so $\lambda \lesssim \xi^{\bar{\mathfrak{x}}}$, resulting in $\lambda \sim \mu$, which contradicts with the assumption that $\lambda \nsim \mu$.

Let $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \gtrsim \mathcal{B}$ as well as $\mathcal{A} \lesssim \mathcal{B}$.

Theorem 11. $\mathcal{A} \sim \mathcal{B}$ *if and only if there exist* $\xi \in \mathcal{A}$ *and* $\zeta \in \mathcal{B}$ *such that* $\xi \sim \zeta$.

Proof. Suppose that $\mathcal{A} \sim \mathcal{B}$. Then there exist $\lambda, \nu \in \mathcal{A}$ and $\mu, \eta \in \mathcal{B}$ such that $\lambda \gtrsim \mu$ and $\nu \lesssim \eta$. Having $\nu = \lambda^{\bar{\mathfrak{x}}}$ for some $\mathfrak{x}$ and $\lambda^{\bar{\mathfrak{x}}} \gtrsim \mu^{\bar{\mathfrak{x}}}$ · it holds that $\mu^{\bar{\mathfrak{x}}} \lesssim \eta$,

8) The number of elements of $\mathfrak{G}$ is finiite.

9) By Theorem 5.

10) $\lambda \gneqq \mu$ symbols the fact that $\lambda \gtrsim \mu$ but $\lambda \nsim \mu$.

while $\mu^{\xi} \gtrsim \eta$ by Theorem 9 ; hence $\lambda^{\xi} \lesssim \mu^{\xi}$. Finally $\lambda^{\xi} \sim \mu^{\xi}$ where, of course, $\lambda^{\xi} \in \mathcal{A}$, $\mu^{\xi} \in \mathcal{B}$. The converse is needless to say.

§ 7. Some Necessary Conditions in Special Cases.

In this paragraph we shall arrange some necessary conditions which are fulfilled by a pair of comparable operations under the special assumptions. If E with the operation λ has a right (left) identity e or right (left) zero[11] n, then for the sake of simplicity we shall say that the operation λ has a right (left) identity e or a right (left) zero n respectively, or say that e or n is a right (left) identity or a right (left) zero of λ respectively.

Ideals [11] $I_\lambda^l(a)$, $I_\lambda^r(a)$ for $a \in E$ are defined as

$$I_\lambda^l(a) = [x \lambda a \mid x \in E], \quad I_\lambda^r(a) = [a \lambda x \mid x \in E].$$

Theorem 12.

(I) *If λ has a right identity and $\lambda \gtrsim \mu$, then $\mathfrak{R}_\mu \subset \mathfrak{R}_\lambda$ and $I_\mu^r(a) \subset I_\lambda^r(a)$ for eve every $a \in E$.*

(II) *If μ has a left identity and $\lambda \gtrsim \mu$, then $\mathfrak{L}_\lambda \subset \mathfrak{L}_\mu$ and $I_\lambda^l(a) \subset I_\mu^l(a)$ for every $a \in E$.*

Proof of (I). By the assumption, there is such an element e that $a \lambda e = a$ for every $a \in E$. Since $\lambda \gtrsim \mu$, we have $a \mu x = (a \lambda e) \mu x = a \lambda (e \mu x)$ for every $x \in E$. From this we get $\mathfrak{R}_\mu \subset \mathfrak{R}_\lambda$ and $I_\mu^r(a) \subset I_\lambda^r(a)$.

Theorem 13.

(I) *The element e is a right identity of λ as well as a left identity of μ. Then either $\lambda = \mu$ or $\lambda \not\sim \mu$.*

(II) *The element e is a left identity of λ as well as a right identity of μ. Then either $\lambda = \mu$ or $\lambda \not\sim \mu$.*

Proof. of (I). Suppose $\lambda \gtrsim \mu$, then $x \mu y = (x \lambda e) \mu y = x \lambda (e \mu y) = x \lambda y$ for every x and $y \in E$. Hence $\lambda = \mu$.

Theorem 14. *If the element e is the two-sided identity of both λ and μ, then either $\lambda = \mu$ or $\lambda \not\approx \mu$.*

Proof. Suppose $\lambda \gtrsim \mu$ or $\lambda \lesssim \mu$, then we have $x \lambda y = x \mu y$ for every x and $y \in E$; hence $\lambda = \mu$.

Theorem 15.

(I) *If $\lambda \gtrsim \mu$ then a right zero of μ implies a right zero of λ.*

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11) By a right zero n of E is meant such an element n that $x \lambda n = n$ for all $x \in E$.

(II) *If* $\lambda \gtrsim \mu$ *then a left zero of* λ *implies a left zero of* μ.

Proof. of (I) Let n be a right zero of μ. $x \lambda n = x \lambda (y \mu n) = (x \lambda y) \mu n = n$.

Theorem 16.

(I) *If* $\lambda \gtrsim \mu$ *and* n *is a right zero of* λ, *then* $n \mu y$ *for* $y \in E$ *is a right zero of* λ.

(II) *If* $\lambda \gtrsim \mu$ *and* n *is a left zero of* μ, *then* $x \lambda n$ *for* $x \in E$ *is a left zero of* μ.

Proof of (I) For every $x \in E$, $x \lambda (n \mu y) = (x \lambda n) \mu y = n \mu y$.

Theorem 17.

(I) *If* $\lambda \gtrsim \mu$ *and* n *is the only right zero of* λ, *then* n *is a left zero of* μ.

(II) *If* $\lambda \gtrsim \mu$ *and* n *is the only left zero of* μ, *then* n *is a right zero of* λ.

Proof of (I) By Theorem 16 (I), $n \mu y$ for $y \in E$ is a right zero of λ. From the uniqueness of right zero follows $n \mu y = n$ for every $y \in E$.

Notes.

[1] The study of semigroups semigroups has been achieved by many mathematcians, Arnold, Lorenzen, Clifford, Suschkewitch, etc., but I have not yet read their works. With respect to the representation of semigroups, see

E. Hille: Functional analysis and semi-groups, 1946, p. 147.

[2] Let P, Q, R be transformations of a set M. By the definition of product,

$$\{(PQ)R\}x = R\{(PQ)x\} = R\{Q(Px)\}, \ \{P(QR)\}x = (QR)(Px) = R\{Q(Px)\},$$

and so $\{(PQ)R\}x = \{P(QR)\}x$ for every $x \in M$. Hence $(PQ)R = P(QR)$.

[3] If $E(\lambda)$ is a semigroup,

$$\{R_\lambda(a) R_\lambda(b)\}x = R_\lambda(b)\{R_\lambda(a)x\} = R_\lambda(b)(x \lambda a) = (x \lambda a) \lambda b$$
$$= x \lambda (a \lambda b) = R_\lambda(a \lambda b) x; \quad \text{therefore} \quad R_\lambda(a) R_\lambda(b) = R_\lambda(a \lambda b).$$

Similarly $L_\lambda(a) L_\lambda(b) = L_\lambda(b \lambda a)$; hence if $E(\lambda)$ is a semigroup, then $\mathfrak{R}_\lambda$ aud $\mathfrak{L}_\lambda$ are algebraic subsystems of $\mathfrak{T}_E$, and consequntly semigroups. However this converse is not true unless, the correspondence $a \leftrightarrow R_\lambda(a)$ is one-to-one.

[4] We suppose that E is a semigroup. Evidently $(x \bar{\lambda} y) \bar{\lambda} z = x \bar{\lambda} (y \bar{\lambda} z)$ for $x, y, z \in E$; by the definition of $\bar{\lambda}$, $(p \bar{\lambda} x) \bar{\lambda} y = p \bar{\lambda} (x \bar{\lambda} y)$, $(x \bar{\lambda} p) \bar{\lambda} y = x \bar{\lambda} (p \bar{\lambda} y)$, and $(x \bar{\lambda} y) \bar{\lambda} p = x \bar{\lambda} (y \bar{\lambda} p)$ for $x, y \in \bar{E}$. Thus $\bar{E}$ is a semigroup. The converse is proved by Lemma 1 and 2.

〔5〕〔6〕 Birkoff : Latfice theory, 1948, *p.* 4.

〔7〕 There are cases that the transitive law holds, even if no universality is assumed. For example,

ξ	a	b
a	a	b
b	a	b

η	a	b
a	a	a
b	a	a

ζ	a	b
a	a	a
b	b	b

where surely $\xi \gtrsim \eta$, $\eta \gtrsim \zeta$ and $\xi \gtrsim \zeta$.

〔8〕 We can prove them not by Theorem 1, but directly by the product tables. In greater detail, Takayuki Tamura : On the condition for semigroup (Japanese), Shikoku Sugaku Danwa, No. 2, 1951.

〔9〕 Furtheremore we have $\lambda \not\equiv \nu$.

〔10〕 In reality it holds that $\mathfrak{A}(\lambda^{\mathfrak{x}})=\mathfrak{A}(\lambda)$ for any $\mathfrak{x} \in \bar{\mathfrak{S}}$.

〔11〕 Takayuki Tamura, Characterization of groupoids and semilattices by idealds in a semigroup, Journal of Science of the Gakugei Faculty Tokushima University, Vol 1, 1950, p. 37.

August 1951,

Gakugei Faculty,
Tokushima University.

Addendum to the paper "On a relation between local convexity and entire convexity." in this Journal, vol. 1.

In p. 25. vol. 1. I defined "convex point x of M", which is explained additionally as following.

If there exists $\delta > 0$ such that $U(x;\varepsilon) \cap M$, as far as non-null, is con ex for any positive $\varepsilon \leqq \delta$, *the point of the space* Ω *is called a convex point regarding* M, *or* M *is said to be (locally) convex at* x.

Furthermore I correct the errors in the same paper as below.

		error	correct
line 4,	page 29,	for any $\varepsilon > 0$	for a sufficiently small $\varepsilon > 0$
last line	page 29,	for some $\gamma < 0$	for some ξ_0, $\gamma > 0$