JOURNAL OF SCIENCE
OF
THE GAKUGEI FACULTY
TOKUSHIMA UNIVERSITY

Volume 1

MATHEMATICS

1950

PUBLISHED
BY
TOKUSHIMA UNIVERSITY
TOKUSHIMA, JAPAN

All communications relating to this publication should be addressed to the Redaction of the Journal of science of the Gakugei Faculty, Tokushima University, Tokushima, Japan.

JOURNAL OF SCIENCE
OF
THE GAKUGEI FACULTY
TOKUSHIMA UNIVERSITY

Volume 1

MATHEMATICS

1950

PUBLISHED
BY
TOKUSHIMA UNIVERSITY
TOKUSHIMA, JAPAN

CONTENTS

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

ON THE BEHAVIOUR OF POWER SERIES ON THE BOUNDARY OF THE SPHERE OF ANALYTICITY IN ABSTRACT SPACES

By

Isae SHIMODA

(Received Dec. 20, 1950)

In classical analysis there exists a singular point at least on the circle of convergence of the power series, but this is not true generally in the case of the power series in complex-Banach-spaces. In this paper we shall investigate a necessary and sufficient condition for power series in complex-Banach-spaces to be analytic at all points on the boundary of the sphere of analyticity.

Let E and E' be two complex-Banach-spaces and an E'-valued function $h_n(x)$ defined on E be a homogeneous polynomial of degree n. Then the radius of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$ exsists, which is written by τ[*]. The sphere $||x||<\tau$ is called the sphere of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$.

Theorem 1. *In order that $\sum_{n=0}^{\infty} h_n(x)$ is analytic at all points on the boundary of the sphere of analyticity, it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} ||h_n(x)||} < \frac{1}{\tau} \tag{1}$$

for an arbitrary compact set G extracted from the set $||x||=1$.

Proof. Let $\sum_{n=0}^{\infty} h_n(x)$ be analytic at all points on $||x||=\tau$. If a compact set G extracted from $||x||=1$ exists which stisfies the following equality

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} ||h_n(x)||} = \frac{1}{\tau},$$

we have

$$\frac{1}{\tau+\varepsilon_i} < \sqrt[n]{\sup_{x\in G} ||h_{n_i}(x)||} \tag{2}$$

*) Isae Shimoda, On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

for a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_n > \dots$, which tends to zero, and n_i which corresponds to ε_i, where $i=1, 2, \dots, n, \dots$. Since G is compact, there exists x_i in G which satisfies

$$\sup_{x \in G} \|h_{n_i}(x)\| = \|h_{n_i}(x_i)\| .$$

Then we can select from $\{x_i\}$ a subsequence which converges to x_0, of course $x_0 \in G$. In order not to change notation we shall suppose simply that the sequence $\{x_i\}$ itself converges to x_0.

Put $(\tau+\varepsilon_i)x_i = y_i$ and $\tau x_0 = y_0$, then y_i converges to y_0. From (2), we have

$$1 < \|h_{n_i}(y_i)\| . \tag{3}$$

where $i=1, 2, 3, \dots$.

Let M be a compact set composed of $y_0 e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Since $\sum_{n=0}^{\infty} h_n(x)$ is analytic on M, we can find a finite system of neighbourhoods U_j of $y_0 e^{i\theta_j}$ $(j=1, 2, \dots, n_0)$ such that $\sum_{j=1}^{n_0} U_j$ covers M and $\|\sum_{n=0}^{\infty} h_n(y)\| \leq N$ for $y \in \sum_{j=1}^{n_0} U_j$. Now we choose two small positive numbers δ and ρ, so that $y\alpha \in \sum_{j=1}^{n_0} U_j$, where $\|y-y_0\| \leq \rho$ and $|\alpha| = 1+\delta$. Then we have

$$\|h_n(y)\| = \left\| \frac{1}{2\pi i} \int_{|\alpha|=1+\delta} \frac{\sum_{n=0}^{\infty} h_n(\alpha y)}{\alpha^{n+1}} d\alpha \right\| \leq \frac{N}{(1+\delta)^n} \tag{4}$$

for $n=1, 2, \dots$ and $\|y-y_0\| < \rho$.

Since y_i converges to y_0, (4) contradicts to (3). This shows that the condition (1) is necessary.

Let y_0 be an arbitrary point on $\|y\| = \tau$. Suppose that there exists a sequence $\{y_n\}$ which converges to y_0 and satisfies the following inequalities

$$\varlimsup_{n \to \infty} \sqrt[n]{\|h_n(y_i)\|} \geq 1-\varepsilon_i \tag{5}$$

for $i=1, 2, \dots$, where a sequence of positive numbers $\{\varepsilon_n\}$ converges to zero with $\varepsilon_{n+1} < \varepsilon_n$. Put $\frac{y_i}{\|y_i\|} = x_i$ and $\{x_i\} = G$. Then G is a compact set on $\|x\| = 1$. Now we assume (1). Then there exists a positive number ε such that $\varlimsup_{n \to \infty} \sqrt[n]{\sup_{x \in G} \|h_n(x)\|} \leq \frac{1}{\tau+3\varepsilon}$. From this, we have $\|h_n(x_i)\| \leq \frac{1}{(\tau+2\varepsilon)^n}$, for $n \geq n_0$ and $i=1, 2, \dots$. Since $x_i = \frac{y_i}{\|y_i\|}$, $\|h_n(y_i)\| \leq \left(\frac{\|y_i\|}{\tau+2\varepsilon}\right)^n$, for $n \geq n_0$ and $i=1, 2, \dots$. On the other hand, there exists N such that $\|y_i\| < \tau+\varepsilon$ for $i \geq N$, because $y_i \to y_0$ and $\|y_0\| = \tau$. Thus we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(y_i)\|}\leq\frac{\|y_i\|}{\tau+2\varepsilon}\leq\frac{\tau+\varepsilon}{\tau+2\varepsilon}<1$$

for $i\geq N$, contradicting to (5). From this we can easily see that there exist two positive numbers δ and ε such that $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(y)\|}\leq 1-\varepsilon$ uniformly for $\|y-y_0\|<\delta$. Hence, $\sum_{n=0}^{\infty} h_n(x)$ is uniformly convergent in $\|y-y_0\|<\delta$ and then $\sum_{n=0}^{\infty} h_n(x)$ is analytic in $\|y-y_i\|<\delta$. This completes the proof.

An example is afforded which is analytic at all points on the boundary of the sphere of analyticity. Put $h_n(x)=\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n$, where $x=(x_1, x_2, \dots)$ is a point of complex-l_2-spaces, and $h_n(x)$ takes complex numbers as its values. Then the radius of analyticity of $\sum_{n=0}^{\infty} h_n(x)$ is 1 and yet $\sum_{n=2}^{\infty} h_n(x)$ is analytic everywhere on $\|x\|=1$.

The radius of analyticty of $\sum_{n=2}^{\infty} h_n(x)$ is given by

$$\frac{1}{\tau}=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\ ^{*)}$$

Since $\|x\|=1$, $|x_i|\leq 1$ for $i=1, 2, \dots$. Therefore

$$\left.\begin{aligned}\frac{1}{\tau}&=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\left|\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n\right|}\\&\leq\overline{\lim_{n\to\infty}}\sqrt[n]{n\left(1-\frac{1}{n}\right)^n}\\&=1\end{aligned}\right\}\qquad(6)$$

Now put $X_m=(0,\dots, 0, 1, 0,\dots)$, where only m-th coordinate is 1 and others are all zero. Since $\|X_m\|=1$, we have

$$\frac{1}{\tau}\geq\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(X_m)\|}=\overline{\lim_{n\to\infty}}\sqrt[n]{\left(1-\frac{1}{m}\right)^n}=1-\frac{1}{m},$$

for $m=2, 3, \dots$.

Hence from (6), we see that $\tau=1$.

Let G be an arbitrary compact set on $\|x\|=1$, then there exists the convergent series of non negative constant $\sum_{n=1}^{\infty} a_n^2=1$ such that $\sum_{n=m}^{\infty}|x_n|^2<\sum_{n=m}^{\infty} a_n^2$ for $x\in G$ and $m=1, 2, 3, \dots$. If $a_1=a_2=a_3=\cdots=a_{n_0}=0$ and $a_{n_0+1}\neq 0$, $|x_m|^2\leq 1$ for $m=1, 2, \dots, n_0+1$ and $|x_m|^2<\sum_{n=m}^{\infty}|x_n|^2<\sum_{n=n_0+2}^{\infty} a_n^2<1$ for

*) Isae Shimoda, On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

$m \geq n_0+2$. Put $\delta = \max\left(1-\frac{1}{n_0+1}, \sqrt{\sum_{n_0+2}^{\infty} a_n^2}\right)$, then $\delta < 1$. Thus we have $\|h_n(x)\| = |\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n| < \sum_{m=2}^{n_0+1}\left(1-\frac{1}{m}\right)^n + \sum_{m=n_0+2}^{n} |x_m|^n < n\delta^n$. Hence, $\overline{\lim}_{n\to\infty} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} \leq \delta < 1$. Thus Theorem 1 is applicable, and we see that $\sum_{n=2}^{\infty} h_n(x)$ is analytic everywhere on the boundary of the sphere of analyticity.

Theorem 2. *In order that there exists at least a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on the boundary of the sphere of analyticity, it is necessary and sufficient to exist at least a compact set G on $\|x\|=1$ which satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau} \tag{7}$$

Proof. If there does not exist a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on $\|x\|=\tau$, it must be analytic on $\|x\|=\tau$. By appealing to Theorem 1, we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} < \frac{1}{\tau},$$

in contradiction to our assumption that $\overline{\lim}_{n\to\infty} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau}$. The inverse is proved as well.

Similarly we have Theorem 3 from Theorem 1.

Theorem 3. *If a power series $\sum_{n=0}^{\infty} h_n(x)$ is analytic at all points on the boundary of its sphere of analyticity $\|x\|=\tau$, then we have*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} < \frac{1}{\tau} \tag{8}$$

for an arbitrary point x on $\|x\|=1$.

From Theorem 3, we have following theorem.

Theorem 4. *If a point x, which lies on $\|x\|=1$, satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} = \frac{1}{\tau},$$

then there exists at least a singular point on $\|x\|=\tau$.

The condition (8) is necessary for $\sum_{n=0}^{\infty} h_n(x)$ to be analytic on the boundary of its sphere of analyticity, but is not sufficient as the following example shows.

Put $h_n(X)=x^{n-1}y$ in the complex-2-dimensional spaces, then $h_n(X)$ is a homogeneous polynomial of degree n, where $X=(x, y)$. We have

$$\sup_{\|X\|=1} \sqrt[n]{\|h_n(X)\|} = \sup_{\|X\|=1} \overline{\lim_{n\to\infty}} |x|^{\frac{n-1}{n}} |y|^{\frac{1}{n}}$$
$$= \sup_{\|X\|=1} |x|$$
$$=1$$

That is, the radius of analyticity of $\sum_{n=1}^{\infty} h_n(X)$ is 1. Now let G be a compact set on $\|X\|=1$ composed of $X_0=(1,0)$ and $X_m=\left(\sqrt{1-\frac{1}{m}}, \sqrt{\frac{1}{m}}\right)$, with $m=1, 2, \dots$.

Then we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{X\in G} \|h_n(X)\|} = \overline{\lim_{n\to\infty}} \sqrt[2n]{\left(1-\frac{1}{n}\right)^{n-1} \frac{1}{n}} = 1,$$

because $(1-t)^{n-1}t$ takes its maximum at $t=\frac{1}{n}$ in the interval $0\leq t\leq 1$. Thus Theorem 2 is applicable and we see that $\sum_{n=1}^{\infty} h_n(X)$ has a singular point on the boundary of its sphere of analyticity.

On the other hand, we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(X)\|} = \overline{\lim_{n\to\infty}} |x|^{\frac{n-1}{n}} |y|^{\frac{1}{n}}$$
$$= |x|<1, \text{ for } y\neq 0 \text{ on } \|X\|=1,$$
$$= 0<1, \quad \text{for } y=0 \text{ on } \|X\|=1.$$

This shows that (8) is satisfied.

Gakugei Faculty, Tokushima University

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

KARAKTERAJ ECOJ DE LINIA KONTIÑUUMO

De

Mitio NAGUMO

(Communicated by Y. Watanabe, Oct. 1, 1950)

Linia kontinuumo signifas simple ordigitan sistemon kiu estas izomorfa je la sistemo de la realaj nombroj. En tiu ĉi noto ni volas montri, ke la linia kontinuumo karakteriĝas kiel kontinua simple ordigita sistemo kun simple transitiva grupo de aŭtomorfismoj.

1. Sistemo S estas *kontinua simple ordigita sistemo* en la sekvanta senco:

(I_1) Se $x_\nu \in S(\nu=1,2)$, tiam ĝuste unu el la tri sekvantaj kazoj okaziĝas: (1) $x_1=x_2$, (2) $x_1<x_2$, (3) $x_2<x_1$.

(I_2) Se $x_1<x_2$ kaj $x_2<x_3$, tiam $x_1<x_3$.

(I_3) (aksiomo de Dedekind) Estu S_1, S_2 partoj de S tiaj ke

$$S_1 \cup S_2 = S\,,\quad S_1 \cap S_2 = \{\}^{1)},$$

kaj $x_1<x_2$ por ĉiuj $x_\nu \in S_\nu(\nu=1,2)$. Tiam ekzistas ĝuste unu $x_0 \in S$ tia ke

$$x_1 \leqq x_0 \leqq x_2 \text{ por ĉiuj } x_\nu \in S_\nu(\nu=1,2).$$

2. *Aŭtomorfismo* φ de S signifas 1-1 bildadon de S sur la saman S tia ke el $x_1<x_2$ rezultas ke $\varphi(x_1)<\varphi(x_2)$. Sistemo $\mathfrak{A}$ de aŭtomorfismoj de S estas *simple transitiva grupo* en la sekvanta senco:

(II_1) Se $\varphi_\nu \in \mathfrak{A}(\nu=1,2)$, tiam $\varphi_2\varphi_1^{-1} \in \mathfrak{A}$.

(II_2) Por ciu paro x_1, x_2 el S ekzistas $\varphi \in \mathfrak{A}$ tia ke $\varphi(x_1)=x_2$.

(II_3) Se $\varphi_1(x_0)=\varphi_2(x_0)$ por iu $x_0 \in S$, tiam $\varphi_1(x)=\varphi_2(x)$ por ĉiu $x \in S$.

Laŭ bone konata maniero el (I) rezultas la ekzisteco de la *supra limo* de iu ajn supre barita parto de S.

TEOREMO. *Se $\varphi_1(x_0)<\varphi_2(x_0)$ por iu $x_0 \in S$, tiam $\varphi_1(x)<\varphi_2(x)$ por ĉiu $x \in S$.*

Por pruvi tion ni bezonas la sekvantan:

---

1) $\{\}$ signifas la malplenan aron.

HELPA TEOREMO. *Se* $\varphi_1(\xi)<\varphi_2(\xi)$ *por iu* $\xi\in S$, *tiam ekzistas* $'\xi$ *kaj* $\xi'(\in S)$ *tiaj ke* $'\xi<\xi<\xi'$ *kaj* $\varphi_1(x)<\varphi_2(x)$ *por* $'\xi<x<\xi'$.

(*Pruvo*) Metu $'\xi=\varphi_2^{-1}\varphi_1(\xi)$ kaj $\xi'=\varphi_1^{-1}\varphi_2(\xi)$, tiam $\varphi_2('\xi)=\varphi_1(\xi)<\varphi_2(\xi)$ kaj $\varphi_1(\xi')=\varphi_2(\xi)>\varphi_1(\xi)$. Sekve $'\xi<\xi<\xi'$. Por $'\xi<x\leqq\xi$ ni havas $\varphi_1(x)\leqq\varphi_1(\xi)=\varphi_2('\xi)<\varphi_2(x)$. Kaj por $\xi\leqq x<\xi'$ ni havas $\varphi_1(x)<\varphi_1(\xi')=\varphi_2(\xi)\leqq\varphi_2(x)$.

(*Pruvo de Teoremo*) Se ekzistus iu $x_1>x_0$ tia ke $\varphi_1(x_1)>\varphi_2(x_1)$, tiam metu $\bar{x}=\sup\{x\,|\,x<x_1,\ \varphi_1(x)<\varphi_2(x)\}(\in S)$.

Do $x_0\leqq\bar{x}\leqq x_1$. Se estus $\varphi_1(\bar{x})\lesseqgtr\varphi_2(\bar{x})$ (sen egaleco), tiam ekzistus, laŭ Helpa Teoremo, intervalo $'\xi<x<\xi'$ tia ke $'\xi<\bar{x}<\xi'$ kaj $\varphi_1(x)\lesseqgtr\varphi_2(x)$ por $'\xi<x<\xi'$. Tio kontraŭdiras kun la difino de $\bar{x}$. Sekve ni havas $\varphi_1(\bar{x})=\varphi_2(\bar{x})$. Kaj el (II_3) rezultas ke $\varphi_1(x)=\varphi_2(x)$ por ĉiu $x\in S$, kio kontraŭdiras kun $\varphi_1(x_0)<\varphi_2(x_0)$. En la simila maniero la ekzisteco de iu $x_1<x_0$ tia ke $\varphi_1(x_1)<\varphi_2(x_1)$ kondukas nin al absurdeco. La teoremo estas do pruvita, ĉar la kazo $\varphi_1(x_1)=\varphi_2(x_1)$ estas ne ebla.

3. Nun elektu iu ajn difinitan elementon el S kaj nomu ĝin "0". Por ĉiu elemento $a\in S$ ekzistas laŭ (II_2) kaj (II_3) ĝuste unu $\varphi_a\in\mathfrak{A}$ tia ke $\varphi_a(0)=a$. Difinu la operacion $a\circ b$ por ĉiu paro a, b el S per

$$a\circ b=\varphi_b(a)\,.$$

Tiam ni havas:

(III_1) $(a\circ b)\circ c=a\circ(b\circ c)$ por ĉiuj $a, b, c\in S$.

(III_2) $0\circ a=a\circ 0=a$ por ĉiu $a\in S$.

(III_3) Por ciu $a\in S$ ekzistas ĝuste unu $\bar{a}$ tia ke $a\circ\bar{a}=\bar{a}\circ a=0$.

(III_4) Se $a_1<a_2(a_\nu\in S)$, tiam por ĉiu $x\in S$ veriĝas

(1) $a_1\circ x<a_2\circ x$, (2) $x\circ a_1<x\circ a_2$.

(III_5) (aksiomo de Arkimedes) Se $0<a$ kaj $0<b$ $(a, b\in S)$, tiam ekzistas natura nombro n tia ke

$$na=\underbrace{a\circ a\circ\cdots\circ a}_{n\text{ foje}}>b.$$

(III_6) $a\circ b=b\circ a$ por ĉiu paro a, b el S (*komutebleco* de $\circ$).

(*Pruvoj*) (III_1) La ambaŭ membroj de la egalaĵo egalas je $\varphi_c\varphi_b\varphi_a(0)$. (III_2) $0\circ a=\varphi_a(0)=a$. Kaj $\varphi_0(x)=x$ laŭ (II_3) ĉar $\varphi_0(0)=0$, sekve $a\circ 0=\varphi_0(a)=a$. (III_3) Metu $\varphi_a^{-1}(0)=\bar{a}$, tiam $a\circ\bar{a}=\varphi_a\varphi_a^{-1}(0)=0$. Kaj $\varphi_{\bar{a}}(x)=\varphi_a^{-1}(x)$ laŭ (II_3), ĉar $\varphi_{\bar{a}}(0)=\varphi_a^{-1}(0)$, sekve $\bar{a}\circ a=\varphi_{\bar{a}}(a)=\varphi_a^{-1}(a)=0$.

(III_4) (1) $a_1\circ x=\varphi_x(a_1)<\varphi_x(a_2)=a_2\circ x$.

(2) El $\varphi_{a_1}(0) < \varphi_{a_2}(0)(\varphi_{a_\nu}(0) = a_\nu)$ rezultas ke $\varphi_{a_1}(x) < \varphi_{a_2}(x)$ laŭ Teoremo, nome $x \circ a_1 < x \circ a_2$.

(III_5) kaj (III_6) sekvas el I_1-I_3 kaj III_1-III_4. Vidu la libron "Koncepto de la nombroj" (japone) de Prof. Takagi, (1949 el Iwanami) p. 92.

Konsekvence ni povas skribi $a+b$ anstataŭ $a \circ b$. La ecoj I_1-I_3 kaj III_1-III_3 montras ke *S estas izomorfa je la sistemo de la realaj nombroj*. Komparu la cititan libron de Prof. Takagi.

Osaka University.

Journal of Science of the Gakugei Faculty,
Tokushima University
vol. 1, 1950

ON LATTICOIDS.

By

Naoki KIMURA

(Communicated by Y. Watanabe, Nov. 20, 1950)

1. G. Birkhoff presented the following problem in his book*) :

Problem 7. What are the consequences of weakening L 1 to

$$x \cup x = x \cap x$$

and L 4 to

$$x \cup (x \cap y) = x \cap (x \cup y) ?$$

In this note we shall discuss the structure of the system on the results of weakening L 1 to

$$x \cup x = x \cap x$$

and L 4 to

$$x \cup (x \cap y) = x \cap (x \cup z).$$

Such a system is called a *latticoid* below.

Types of latticoids are seemed to be very complicated, and we cannot yet determine them, but in the special case (called simple) we shall give all types by means of the corresponding lattice (denoted as $\sigma(L)$) and a set of cardinal numbers.

Latticoids, above all, simple latticoids are seemed to be the most natural generalization of lattices, with respect to several aspects.

2. A set L of elements a, b, c, ... which satisfies the following five conditions, is called a *latticoid* :

(0) Two binary operations $\cup$ and $\cap$ are defined to each ordered pairs a, b of L:

$$a, b \in L \quad \text{imply} \quad a \cup b \in L \text{ and } a \cap b \in L,$$

(1) $a \cup a = a \cap a$,

(2) $a \cup b = b \cup a$ and $a \cap b = b \cap a$,

*) G. Birkhoff: Lattice Theory p. 18, 1948.

(3) $(a\cup b)\cup c=a\cup(b\cup c)$, and $(a\cap b)\cap c=a\cap(b\cap c)$,

(4) $a\cup(a\cap b)=a\cap(a\cup c)$.

Note that the last condition (4) means two elements $a\cup(a\cap b)$ and $a\cap(a\cup b)$ are always equal, and do not depend upon b.

Put

$$\rho(a)=a\cup a=a\cap a,$$
$$\sigma(a)=a\cup(a\cap b)=a\cap(a\cup c).$$

Lemma 1.

(1) $\sigma(a)=a\asymp\rho(a)=a\cup a\cup a=a\cap a\cap a=a\cup(a\cap a)=a\cap(a\cup a)$.

(2) *Let $p(a)$ be a polynomial of a of degree n, which is greater than or equal to 3, then we have*

$$p(a)=\sigma(a),$$

(3) $a\asymp\sigma(a)=\sigma(a)$,

(4) $\sigma(a)\asymp\sigma(a)=\sigma(a)$,

(5) $\sigma(\sigma(a))=\sigma(a)$,

(6) $\sigma(a\asymp b)=\sigma(a)\asymp\sigma(b)=\sigma(a)\asymp b=a\asymp\sigma(b)$.

(3), (4) *and* (5) *are the special cases of* (2).

Proof.

(1) By the definition of $\rho(a)$ and $\sigma(a)$,

(3) $a\cup\sigma(a)=a\cup(a\cap(a\cup a))=\sigma(a)$ (the definition of $\sigma(a)$).

Dually we have
$$a\cap\sigma(a)=\sigma(a),$$

(2) By the (n−4) iterations of (3).

(4) and (5) are only special cases of (2).

(6) $\sigma(a\cup b)=(a\cup b)\cup((a\cup b)\cap b)=a\cup b\cup\sigma(b)=a\cup\sigma(b)$.

In the same way, we have

$$\sigma(a\cup b)=\sigma(a)\cup b,$$

and,

$$\begin{aligned}\sigma(a\cup b)&=\sigma(\sigma(a\cup b))\quad\text{(By (5).)}\\&=\sigma(a\cup\sigma(b))\\&=\sigma(a)\cup\sigma(b).\end{aligned}$$

Hence

$$\sigma(a\cup b)=\sigma(a)\cup\sigma(b)=\sigma(a)\cup b=a\cup\sigma(b).$$

Dually, we have

$$\sigma(a\cap b)=\sigma(a)\cap\sigma(b)=\sigma(a)\cap b=a\cap\sigma(b).$$

Theorem 1. *The sublatticoid $\sigma(L)$ of L is a lattice, where*

$$\sigma(L) = (\sigma(a)\,;\ a \in L).$$

Proof. $\sigma(L)$ is a subset of L, so $\sigma(L)$ satisfies the conditions (2) and (3) of the lattice. The preceding lemma shows that $\sigma(L)$ also satisfies the another conditions (0), (1) and (4) of the lattice.

Theorem 1′. *$\sigma(L)$ is the greatest lattice of all sublatticoids of L.*

Proof. If L' is a lattice contained in L, then $\sigma(L')=L'$. Now $L'\subseteq L$ implies $\sigma(L')\subseteq\sigma(L)$. Hence $L'=\sigma(L')\subseteq\sigma(L)$.

Theorem 1″. *A latticoid L is a lattice, if and only if*

$$L = \sigma(L).$$

Proof. If $L=\sigma(L)$, then L is a lattice (Theorem 1.). Conversely, if $L\neq\sigma(L)$, then L cannot be a lattice, for $\sigma(L)$ is the greatest lattice contained in L (Theorem 1′).

Theorem 2. *The mapping $\sigma: a\rightarrow\sigma(a)$ of L onto $\sigma(L)$ (or into L) is a lattice homomorphism in the sense that*

$$\sigma(a \curlywedge b) = \sigma(a) \curlywedge \sigma(b).$$

Proof. By the lemma 1, (6).

The mapping σ yields a partition of L, such that a and b belong to the same class, if and only if $\sigma(a)=\sigma(b)$.

In the partition by the mapping σ, we shall denote the class which contains $a \in L$ as $\bar{a}$ i. e.,

$$\bar{a} = (x\,;\ \sigma(x) = \sigma(a),\ x \in L).$$

We can easily see that

$$L \supseteq \rho(L) \supseteq \rho^2(L) = \sigma(L),$$

where

$$\rho^2(L) = \rho(\rho(L)).$$

Theorem 3. *Let a latticoid L be lattice homomorphic with a lattice L'. Then the lattice $\sigma(L)$ is lattice homomorphic with L'.*

Proof. Let f be the lattice homomorphic mapping of a latticoid L onto a lattice L'. Then we have

$$f(\sigma(a) \curlywedge \sigma(b)) = f(\sigma(a)) \curlywedge f(\sigma(b))\,,$$

for $\sigma(L)$ is a subset of L.

Hence f gives a lattice homomorphism of $\sigma(L)$ with L'.

Theorem 3′. *Let a latticoid L be lattice isomorphic with a lattice L'. Then L must be a lattice which is isomorphic with L'.*

Proof. $\sigma(L)$ is lattice homomorphic with L' by the Theorem 3. And this homomorphism must be a one-to-one mapping. This means that $\sigma(L)$ is lattice isomorphic with L'. But L is lattice isomorphic with L'. Then L is lattice isomorphic with $\sigma(L)$ by the mapping $a \to \sigma(a)$. Hence $L=\sigma(L)$. Therefore by the Theorem 1″ we can conclude that L is a lattice which is isomorphic with L'.

Theorem 4.

$$x \cup a = x \quad \textit{implies} \quad x = \sigma(x),$$

and dually

$$x \cap a = x \quad \textit{implies} \quad x = \sigma(x).$$

Proof. If

$$x \cup a = x,$$

then,

$$x = x \cup a = x \cup a \cup a = x \cup a \cup a \cup a = x \cup \sigma(a) = \sigma(x \cup a) = \sigma(x),$$

for

$$x \cup a = x.$$

In the same way, we have $x=\sigma(x)$, when $x \cap a = x$.

3. A latticoid L will be called *simple*, if for any two elements $a, b \in L$, always $a \asymp b \in \sigma(L)$.

Lemma 2. *If a latticoid L is simple, then*

$$a \asymp b = \sigma(a) \asymp b = a \asymp \sigma(b) = \sigma(a) \asymp \sigma(b) = \sigma(a \asymp b).$$

A latticoid L will be called *latticoid homomorphic* with a latticoid L', if there exists a mapping f of L onto L', such that

$$f(\sigma(a \asymp b)) = \sigma(f(a) \asymp f(b)).$$

In this case f is called a latticoid homomorphism of L with L'. If f is a one-to-one mapping, then the term homomorphism is replaced by isomorphism.

Theorem 5. *For any latticoid L, there exists a simple latticoid L', with which L is latticoid isomorphic.*

Proof. A slight modification of definitions of $\cup$ and $\cap$ of L, such that

$$a \vee b = \sigma(a \cup b),$$

and

$$a \wedge b = \sigma(a \cap b)$$

yields a new latticoid L' with two operations $\vee$ and $\wedge$. It is easy to that L is latticoid isomorphic with L', and L' is a simple latticoid.

Theorem 6. *If a latticoid L is latticoid isomorphic with both simple latticoids L' and L'', then L' and L'' are lattice isomorphic with each other.*

Proof. If L is latticoid isomorphic with both L' and L'', by mappings f_1 and f_2, we have for each element pair $a, b \in L$,

$$\begin{aligned} f_j(a) \vee\!\!\!\wedge f_j(b) &= \sigma(f_j(a) \vee\!\!\!\wedge f_j(b)) && \text{(For, } L' \text{ and } L'' \text{ are both simple.)} \\ &= f_j(\sigma(a) \vee\!\!\!\wedge \sigma(b)) \\ &= f_j(\sigma(a \vee\!\!\!\wedge b)) && (j=1,2) \end{aligned}$$

Hence the mapping $g = f_2 f_1^{-1} : f_1(a) \to f_2(a)$, $a \in L$, gives a lattice isomorphism of L' with L'', that is, L' and L'' are lattice isomorphic with each other.

Remark. It can easily be led by the above theorem, that the two notions, lattice isomorphism and latticoid isomorphism, are coincides, so far as we shall concern with simple latticoids.

A *multiplicity* $\mathfrak{m}_a$ of an element a of L is the cardinal number of the class $\bar{a}$ of L.

Theorem 7. *If any lattice L, and a set of cardinal number $\mathfrak{m}_a$ corresponding to each element $a \in L$ are given, there exists a simple latticoid L', such that $\sigma(L')$ is lattice isomorphic with L and the multiplicity of each element $\sigma(a') \in \sigma(L')$ is $\mathfrak{m}_a$ corresponding to a, where a is a lattice isomorphic image of $\sigma(a')$.*

Proof. Take any element a of L, and construct a set $\bar{a}$, which contains a, and has cardinal number $\mathfrak{m}_a$.

Suppose $\bar{a}$ and $\bar{b}$ have no intersection, if $a \neq b$, and let L' be the set union of $\bar{a}$ for all a of L.

Then L' forms a simple latticoid with operations

$$x \vee\!\!\!\wedge y = a \vee\!\!\!\wedge b,$$

where

$$x \in \bar{a}, \quad y \in b,$$

and

$$L' \supseteq \sigma(L') = L.$$

It is obvious that the multiplicity of each element $x \in \bar{a}$, is $\mathfrak{m}_a$.

Theorem 8. *A simple latticoid L is determined up to lattice isomorphism by means of the lattice $\sigma(L)$ and a set of multiplicity of each element of $\sigma(L)$.*

Proof. Let L_1 and L_2 be two simple latticoids, and $\sigma(L_1)$ and $\sigma(L_2)$ be lattice isomorphic with each other. Moreover let the multiplicity of each element of $\sigma(L_1)$ and that of corresponding element of $\sigma(L_2)$ be the same. The assumption of this theorem enables us to extend the lattice isomorphic mapping between $\sigma(L_1)$ and $\sigma(L_2)$ to a lattice isomorphic mapping between whole L_1 and L_2, naturally:

if $a_1 \leftrightarrow a_2$, $a_1 \in \sigma(L_1)$, $a_2 \in \sigma(L_2)$,
then the cardinal number of $\bar{a}_1$ *and* $\bar{a}_2$ *are the same.*
Thus we can construct a one-to-one mapping between $\bar{a}_1$ *and* $\bar{a}_2$ *so a: to* a_1 *correspond to* a_2.

This extended mapping between L_1 and L_2 must be a lattice isomorphism between them:

if $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$, *then* $\sigma(x_1) \leftrightarrow \sigma(x_2)$, $\sigma(y_1) \leftrightarrow \sigma(y_2)$.
Therefore $x_1 \asymp y_1 = \sigma(x_1) \asymp \sigma(y_1) \leftrightarrow \sigma(x_2) \asymp \sigma(y_2) = x_2 \asymp y_2$.

A slight modification of the proof of the preceding theorem leads the following.

Theorem 9. *A latticoid L is completely determined up to latticoid isomorphism by means of the lattice* $\sigma(L)$ *and a set of multiplicity of each element of* $\sigma(L)$.

(Tokyo Institute of Technology)

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

UNBIASED ESTIMATE OF THE MEAN ABSOLUTE DEVIATION

By

Yoshikatsu WATANABE

(Received Sept. 15, 1950)

If the observed values of a quantity z be $z_1, \dots, z_n$ and the mean value $\bar{z}=\sum z_j/n$, then the deviations are $z_j-\bar{z}=x_j$, $j=1, \dots, n$. The unbiased estimate of the mean square deviation is given by the well-known Bessel's formula

$$\hat{\sigma}^2 = \sum x_j^2/(n-1). \tag{1}$$

However its demonstrations are found hardly legitimate in classical books on least squares, except some fews, e.g. A. F. Craig's elegant proof given in Bulletin of the American Math. Soc., 1936, vol. 42. He pointed out that (1) means nothing but the expectation of the sum of squares

$$\sum x_j^2 = V, \quad \text{i.e.} \quad E(V) = (n-1)\sigma^2, \tag{2}$$

and proved (2) under the assumption that x distributes normally. In the present note a similar process is applied to generalize Peters' formula in regard to the mean absolute deviation ϑ

$$\hat{\vartheta} = \sum |x_j| / \sqrt{n(n-1)}.$$

1° Characteristic. As well known, the distribution function (density of probability) $f(x)$ as well as its characteristic $g(t)$ are defined as follows[1)]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} g(t)dt, \quad g(t) = \int_{-\infty}^{\infty} e^{ixt} f(x)dx. \tag{3}$$

Or, in case of many variables,

$$\left.\begin{aligned} f(x_1, \dots, x_m) &= \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-L} g(t_1, \dots, t_m)dt_1 \dots dt_m, \\ g(t_1, \dots, t_m) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{L} f(x_1, \dots, x_m)dx_1 \dots dx_m, \\ \text{where} \quad L &= ix_1t_1 + \cdots + ix_mt_m. \end{aligned}\right\} \tag{4}$$

1) See the annexed References.

Now, for a single valued continuous function $u=U(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are assumed to be independent, the distribution function $F(u)$ shall be given by

$$F(u)du=\int_D \cdots \int f(x_1, \dots, x_n)dx_1 \dots dx_n \ ,$$

where $x_1, \dots, x_n$ should be taken over the domain D that satisfies the inequality $u \leqq U(x_1, \dots, x_n) \leqq u+du$. To avoid this inconvenience, let us multiply, after Cauchy's devise, both sides by the function[2)]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_u^{u+du} e^{it(U-\xi)} d\xi \ ,$$

which becomes $=1$ in $\langle u, u+du \rangle$, and otherwise $=0$. Then the domain of integration can be extended to the whole n-dimensional space R_n, so

that $$F(u)du=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_u^{u+du} e^{-it\xi} d\xi \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{itU} f(x_1, \dots, x_n)dx_1 \dots dx_n \ .$$

Or putting the inner integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{itU} f(x_1, \dots, x_n)dx_1 \dots dx_n = G(t) \ , \tag{5}$$

we get

$$F(u)du \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[e^{-itu}du \right] G(t)dt \ ,$$

hence

$$F(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu} G(t)dt \ . \tag{6}$$

Thus $G(t)$ gives the characteristic of $F(u)$. Furthermore letting

$$u_k = U_k(x_1, \dots, x_n) \ , \quad k=1, 2, \dots, m, \quad (m<n) \ ,$$

the distribution function $F(u_1, \dots, u_m)$ shall be defined by

$$F du_1 \dots du_m = \int_D \cdots \int f(x_1, \dots, x_n)dx_1 \dots dx_n \ ,$$

where D denotes such a domain that $u_k \leqq U_k \leqq u_k+du_k$, $k=1, \dots, m$. Here again repeating Cauchy's devises m times, and putting as the characteristic

$$G(t_1, \dots, t_m)=\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\vartheta} f(x_1, \dots, x_n)dx_1 \dots dx_n \ , \tag{7}$$

---

2) Cf. the annexed References (*).

where $\theta = it_1U_1 + it_2U_2 + \cdots + it_mU_m$, we obtain

$$F(u_1, \ldots, u_m) = \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\theta}G(t_1, \ldots, t_m)dt_1 \ldots dt_m . \tag{8}$$

2° In our case, assuming that x distributes normally

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{x^2}{2\sigma^2}\right\} \tag{9}$$

under the conditions that

$$u_k = \sum_{\nu=1}^{n} a_{k\nu}x^{\nu}, \quad k = 1, \ldots, m, \text{ and } u_{m+1}(=u) = \sum_{\nu=1}^{n}|x_\nu| , \tag{10}$$

where $1 \leq m \leq n$, and all the coefficients are real, the characteristic in accordance with (7) is given by

$$G(t_1, \ldots, t_{m+1}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{it_1u_1 + \cdots + it_{m+1}u_{m+1}\right\}f(x_1, \ldots, x_n)dx_1 \ldots dx_n ,$$

where f is determined from (9) to be

$$f(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-V/2\sigma^2\right\}, \quad V = \sum_{\nu=1}^{n} x_\nu^2 .$$

Hence we obtain

$$\left.\begin{aligned} G(t_1, \ldots, t_{m+1}) &= \frac{1}{(\sqrt{2\pi}\sigma)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\Theta}dx_1 \ldots dx_n , \\ \text{where} \quad \Theta &= i\sum_{l=1}^{m+1} t_l u_l - V/2\sigma^2 . \end{aligned}\right\} \tag{11}$$

In order to perform the integration, we write

$$\begin{aligned} \Theta = &-\frac{1}{2\sigma^2}\sum_{x_\nu>0}{}' \left\{\left[x_\nu - i\sigma^2(s_\nu + t_{m+1})\right]^2 + \sigma^4(s_\nu + t_{m+1})^2\right\} \\ &-\frac{1}{2\sigma^2}\sum_{x_\nu<0}{}'' \left\{\left[x_\nu - i\sigma^2(s_\nu - t_{m+1})\right]^2 + \sigma^4(s_\nu - t_{m+1})^2\right\} \end{aligned} \tag{12}$$

where

$$s_\nu = \sum_{l=1}^{m} a_{l\nu}t_\nu . \tag{13}$$

So that (11) becomes

$$G(t_1, \ldots, t_{m+1}) = \prod_{\nu=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\left[I_\nu \exp\left\{-\frac{\sigma^2}{2}(s_\nu + t_{m+1})^2\right\} + J_\nu \exp\left\{-\frac{\sigma^2}{2}(s_\nu - t_{m+1})^2\right\}\right], \tag{14}$$

where

$$\begin{matrix} I_\nu \\ J_\nu \end{matrix} = \int_0^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\left[x_\nu - i\sigma^2(s_\nu \pm t_{m+1})\right]^2\right\}dx_\nu . \tag{15}$$

The integrals (15) can be found by utilizing Cauchy's integral theorem to be

$$\sqrt{\frac{\pi}{2}}\sigma \pm i\sigma\int_0^{\sigma(s_\nu \pm t_{m+1})} e^{t^2/2}dt ,$$

so that (14) may be transformed into

$$G(t_1, \dots , t_{m+1}) = \prod_{\nu=1}^{n} \exp\left\{-\frac{\sigma^2}{2}(s_\nu^2 + t_{m+1})\right\}\Big[\cosh(\sigma^2 s_\nu t_{m+1}) + \frac{i}{\sqrt{2\pi}}\Big(\exp\left(-\frac{1}{2}\sigma^2 s_\nu t_{m+1}\right)\int_0^{\sigma(s_\nu + t_{m+1})} e^{t^2/2}dt - \exp\left(-\frac{1}{2}\sigma^2 s_\nu t_{m+1}\right)\int_0^{\sigma(s_\nu - t_{m+1})} e^{t^2/2}dt\Big)\Big] . \quad (16)$$

In particular letting $t_{m+1}=0$, we obtain as the characteristic of the combined distributions function $\Phi(u_1, \dots , u_m)$,

$$G(t_1, \dots , t_m, 0) = \prod_{\nu=1}^{n} \exp\left\{-\frac{\sigma^2}{2}s_\nu^2\right\} \equiv \exp\left\{-\frac{\sigma^2}{2}Q\right\} . \quad (17)$$

Since the numbers considered are all real

$$Q = \sum_{\nu=1}^{n} s_\nu^2 = \sum_{k,l=1}^{m}\sum_{\nu=1}^{n} a_{k\nu}a_{l\nu}t_k t_l = \sum_{k,l} b_{kl}t_k t_l$$

is a positive definite Hermite form (Bt, t), so that the matrix $B=(b_{kl})$ can be transformed into a diagonal one Λ by taking an adequate orthogonal matrix $C=(c_{kl})$ so as $C'BC=\Lambda$: or in other words, by the orthogonal transformation $t=Cz$: $t_l=\sum_k c_{lk}z_k$, we obtain

$$Q = \sum_{l=1}^{m} \lambda_l z_l^2 \ (>0 \text{ except when all } z_l = 0) ,$$

where the coefficients are all >0, and the jacobian $J=\frac{\partial(t_1, \dots , t_m)}{\partial(z_1, \dots , z_m)}=1$. Thus we get

$$G(t_1, \dots , t_m, 0) = \exp\left\{-\frac{\sigma^2}{2}\sum_{l=1}^{m}\lambda_l z_l^2\right\} , \quad (18)$$

and by (8) the corresponding distribution function becomes

$$\Phi(u_1, \dots , u_m) = \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-L-\frac{\sigma^2}{2}Q\right\}dt_1 \dots dt_m$$
$$= \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-L-\frac{\sigma^2}{2}Q\right\}dz_1 \dots dz_m , \quad (19)$$

where $\quad L = i\sum_{l=1}^{m} t_l u_l = i\sum_l u_l \sum_k c_{lk}z_k = i\sum_k v_k z_k ,\quad$ and $\quad v_k = \sum_l c_{lk}u_l$.

Now that the multiple integral in (19) might be decomposed into a product of the form

$$\prod_{l=1}^{m}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-\frac{\sigma^2}{2}\left(\lambda_l z_l^2 + \frac{2i}{\sigma^2}v_l z_l\right)\right\}dz_l , \quad (20)$$

the integration could be performed by availing Cauchy's integral theorem, and we get finally

$$\Phi(u_1, \dots, u_m) = \frac{1}{\sqrt{\lambda_1 \dots \lambda_m}(\sqrt{2\pi}\sigma)^m} \exp\left\{-\frac{1}{2\sigma^2}\sum_{l=1}^{m}\left(\sum_{k=1}^{m} c_{lk}u_k\right)^2 / \lambda_l\right\}. \tag{21}$$

In particular, if $u_1 = \cdots = u_m = 0$, we have

$$\Phi(0, \dots, 0) = 1/\sqrt{\lambda_1 \dots \lambda_m}\,(\sqrt{2\pi}\sigma)^m . \tag{22}$$

3° The expectation of $\sum|x_\nu| = u$. Let $F(u_1, \dots, u_m, u)$ be the combined distribution function of $u = \sum|x_\nu|$, and $u_k (k=1, \dots, m)$ given in (10), and let us find the expectation of u, when $u_1, \dots, u_m$ are assumed to be fest. Here, since the compound probability for $u_1, \dots, u_m$ is $\Phi(u_1, \dots, u_m)\, du_1 \dots du_m$, while the compound probability for $u_1, \dots, u_m$ and u is $F(u_1, \dots, u_m, u)\, du_1 \dots du_m du$, the relative probability becomes F/Φ, and accordingly the required expectation can be given by

$$\bar{u} = \int u \frac{F}{\Phi} du , \tag{23}$$

where the integration should be extended over all the possible values of $u (\geqq 0)$ as far as $u_1, \dots, u_m$ preserve the given fest values. But in virtue of (4), the characteristic $G(t_1, \dots, t_{m+1})$ of $F(u_1, \dots, u_m, u)$ is given by

$$G(t_1, \dots, t_m, t_{m+1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{L + iut_{m+1}\right\} F du_1 \dots du_m du ,$$

where $L = i\sum_{l=1}^{m} t_l u_l$. Whence we get

$$\left(\frac{\partial G}{\partial t_{m+1}}\right)_{t_{m+1}=0} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} iue^L \frac{F}{\Phi} \Phi du du_1 \dots du_m ,$$

which becomes after substitution of (23)

$$\left(\frac{\partial G}{\partial t_{m+1}}\right)_0 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} i\bar{u}e^L \Phi du_1 \dots du_m .$$

Therefore inversely we get by (4)

$$i\bar{u}\Phi = \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-L}\left(\frac{\partial G}{\partial t_{m+1}}\right)_0 dt_1 \dots dt_m . \tag{24}$$

On the other hand we have by (16)

$$G = \prod_{\nu=1}^{n} g_\nu(t_{m+1}) ,$$

where $g_\nu(t_{m+1})$ denotes the ν-th factor in (16), and consequently

$$g_\nu(0)=\exp\left\{-\frac{\sigma^2}{2}s_\nu^2\right\}, \qquad s_\nu=\sum_{l=1}^{m}a_{l\nu}t_l ,$$

and
$$g'_\nu(0)=i\sigma\sqrt{\frac{2}{\pi}}\left[1-\sigma s_\nu\exp\left(-\frac{\sigma^2}{2}s_\nu^2\right)\int_0^{\sigma s_\nu}e^{t^2/2}dt\right].$$

$$\therefore\quad \left(\frac{\partial G}{\partial t_{m+1}}\right)_0=\sum_{\nu=1}^{n}g'_\nu(0)\exp\left(-\frac{\sigma^2}{2}\sum_{\mu\neq\nu}^{n}s_\mu^2\right).$$

This value being substituted in (24), we ought to integrate it, which is somewhat troublesome. However if the first m conditions of (10) be linear homogeneous, that is, if $u_1=\cdots=u_m=0$, then we can perform further integrations. Really, form the result just obtained together with (22), we get

$$\bar{u}=\frac{\sqrt{\lambda_1\ldots\lambda_m}\,\sigma^{m+1}}{\sqrt{2\pi}^{m}}\sqrt{\frac{2}{\pi}}\sum_{\nu=1}^{n}\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\frac{\sigma^2}{2}\left(Q-s_\nu^2\right)\right]dt_1\ldots dt_m\right.$$
$$\left.-\sigma\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2}{2}Q\right)s_\nu\int_0^{\sigma s_\nu}e^{t^2/2}dt_1\ldots dt_m\right\}, \tag{25}$$

Let us evaluate the multiple integrals in the above large bracket, (i) and (ii) say. In order to facilitate the integral (i), we must adopt another orthogonal transformation $t=C_\nu\zeta$, so that the expression in the exponent, which is also a positive definite Hermite form, becomes

$$Q_\nu\equiv Q-s_\nu^2=\sum_{\mu\neq\nu}s_\mu^2=\sum_{k,l}^{m}\left(\sum_{\mu\neq\nu}^{n}a_{k\mu}a_{l\mu}\right)t_kt_l=\sum_l\lambda_{l\nu}\zeta_l^2 ,$$

and after integrations we get

$$(\mathrm{i})=\left(\frac{\sqrt{2\pi}}{\sigma}\right)^m\Big/\sqrt{\lambda_{1\nu}\ldots\lambda_{m\nu}}\,. \tag{26}$$

In regard to (ii), we utilize at first the before mentioned transformation $t=Cz$, so that $Q=\sum_l\lambda_lz_l^2$, and $s_\nu=\sum_l a_{l\nu}t_l=\sum_l d_{\nu l}z_l$, where $d_{\nu l}$ denotes the νl-element of the matrix $d=a'c$. Integrating by parts, we obtain

$$(\mathrm{ii})=\sigma\sum_{l=1}^{m}\frac{d_{\nu l}^2}{\lambda_l}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\frac{\sigma^2}{2}\left[\sum_{k=1}^{m}\lambda_kz_k^2-\left(\sum_{k=1}^{m}d_{\nu k}z_k\right)^2\right]\right\}dz_1\ldots dz_m .$$

Here the expression in the exponent becomes

$$Q_\nu^*\equiv\sum_{k,l}^{m}(\delta_{kl}\sqrt{\lambda_k\lambda_l}-d_{\nu k}d_{\nu l})z_kz_l \qquad (\delta_{kl}=\text{Kronecker's delta})$$
$$=\sum_{l=1}^{m}\lambda_{l\nu}^{*}\zeta_l^{*2}$$

by a third orthogonal transformation $z=C_\nu^*\zeta^*$, and after integration, we get

$$(\text{ii}) = \frac{\sqrt{2\pi}^m}{\sigma^{m+1}} \sum_{l=1}^{m} \frac{d_{\nu l}^2}{\lambda_l \sqrt{\lambda_{1\nu}^* \dots \lambda_{m\nu}^*}} \, . \tag{27}$$

Substituting (26) and (27) in (25), we get, as the generalized Peters' formula,

$$\bar{u} = \sqrt{\lambda_1 \dots \lambda_m} \sqrt{\frac{2}{\pi}} \, \sigma \sum_{\nu=1}^{n} \left[\frac{1}{\sqrt{\lambda_{1\nu} \dots \lambda_{m\nu}}} - \sum_{l=1}^{m} \frac{d_{\nu l}^2}{\lambda_l \sqrt{\lambda_{1\nu}^* \dots \lambda_{m\nu}^*}} \right] . \tag{28}$$

Specially if $m=1$, we obtain (in omitting the suffix 1)

$$\lambda = \sum_{\nu=1}^{n} a_\nu^2 \, , \qquad \lambda_\nu = \sum_{\mu \neq \nu}^{n} a_\mu^2 = \lambda - a_\nu^2 \, , \qquad \lambda_\nu^* = \lambda_\nu \quad (\because \; d_\nu = a_\nu) \, ,$$

and hence

$$\bar{u} = \sqrt{\frac{2}{\pi}} \, \sigma \sum_{\nu=1}^{n} \sqrt{\frac{\lambda - a_\nu^2}{\lambda}} = \sqrt{\frac{2}{\pi}} \, \sigma \sum_{\nu=1}^{n} \sqrt{\sum_{\mu \neq \nu}^{n} a_\mu^2 \Big/ \sum_{\nu=1}^{n} a_\nu^2} \, . \tag{29}$$

More specially in case that all $a_\nu = 1$, i.e. $u_1 \equiv \sum x_\nu = 0$, as in the case of the residual sum of least squares, we have $\lambda = n$, and

$$\bar{u} = \sqrt{\frac{2}{\pi}} \, \sigma \sqrt{n(n-1)} \, .$$

But the absolute mean is

$$\vartheta = \frac{2}{\sqrt{2\pi}\sigma} \int_0^\infty x \exp\left\{ -\frac{x^2}{2\sigma^2} \right\} dx = \sqrt{\frac{2}{\pi}} \, \sigma \, ,$$

so that we get $\bar{u} = \vartheta \sqrt{n(n-1)}$, which means that

$$\hat{\vartheta} = \frac{u}{\sqrt{n(n-1)}} = \frac{\sum |x_\nu|}{\sqrt{n(n-1)}} \, , \tag{30}$$

the so-called Peters' formula.

Usually the unbiased estimate of σ conveniently calculated from Bessel's formula $\hat{\sigma^2} = \sum x_\nu^2/(n-1)$, to be $\hat{\sigma} = \sqrt{\sum x_\nu^2/(n-1)}$. However this is not correct, because $\sqrt{\hat{\sigma^2}} \neq \hat{\sigma}$. It will be rather reasonable to avail the above obtained result $\bar{u} = \sqrt{\frac{2}{\pi}} \, \sigma \sqrt{n(n-1)}$, and to put

$$\hat{\sigma} = \sqrt{\frac{\pi}{2}} \frac{\sum |x_\nu|}{\sqrt{n(n-1)}} \, . \tag{31}$$

Gakugei Faculty, Tokushima University.

References

According to Fourier's integral-theorem, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}e^{it(\xi-x)}f(\xi)d\xi\ , \qquad i=\sqrt{-1}\ .$$

This follows readily from Fourier's expansion

$$f(x)=\frac{1}{l}\sum_{n=0}^{\infty}\int_{-l}^{l}f(\xi)\cos\frac{n\pi}{l}(\xi-x)d\xi$$

by writing $\frac{n\pi}{l}=t$, $\frac{\pi}{l}=dt$, and making $n, l\to\infty$,

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}dt\int_{-\infty}^{\infty}f(\xi)\cos t(\xi-x)d\xi\ .$$

(cf. e.g. Prof. Takagi's Treatise on Analysis, p. 336).
Or

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}f(\xi)e^{it(\xi-x)}d\xi=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixt}dt\int_{-\infty}^{\infty}e^{it\xi}f(\xi)d\xi\ ,$$

whence the relations (3) immediately follow.

Specially, if in $a\leqq x\leqq b$, $f(x)=1$, and otherwise $f(x)=0$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}e^{it(x-\xi)}d\xi \begin{cases} =1\ , & \text{in } a\leqq x\leqq b\ , \\ =0\ , & \text{otherwise.}\end{cases} \qquad (*)$$

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

ON A RELATION BETWEEN LOCAL CONVEXITY AND ENTIRE CONVEXITY

By

Takayuki TAMURA

(Received Sept. 28, 1950)

1. Introduction.

The space considered here is a separable real-Banach space, written Ω. Let us denote points of Ω by $a, b, x, \ldots$ etc.; sets by $M, E, \ldots$ etc.; and real numbers by $\alpha, \lambda, \ldots$ etc.. If there exists $\delta > 0$ such that $U(x;\varepsilon) \frown M$,[1] as far as non-null, is convex for any positive $\varepsilon \leq \delta$, the point x is called a *convex point* of M, or M is said to be (locally) convex at x. As M is always convex at its interior points, the concept of local convexity at x is of special significance in the case x is a boundary point of M, and so the convex point of M implies an interior point or boundary point of M. When M is locally convex everywhere at the boundary, M is said to be locally convex.

Clearly, although convexity (in large) implies local convexity, the converse is not true. In this paper we impose on the local convex set M the condition of its arcwise connectedness, by which we mean that every two points of M can be joined by an arc[2] lying in M. And yet the convexity of M does not necessarily follow, but does that of the interior M^i of M, that is, we get the following result.

Theorem. *If M is locally convex and arcwise connected, then M^i is convex.*

2. Preliminaries.

The symbols $\{x, y\}$, $[x, y]$, etc. are defined as following.
Letting $z(\lambda) = (1-\lambda)x + \lambda y$ for $x \neq y$,

---

1) By $U(x;\varepsilon)$ we mean ε-neighborhood of x, that is,
$E[z \mid \| z - x \| < \varepsilon,\ z \in \Omega]$

2) The set C is called an *arc* if it is homeomorphic with the unit closed interval $[0, 1]$.

$$\{x, y\} = E\left[z(\lambda) \,\middle|\, \begin{matrix} \textit{for all real} \\ \textit{numbers } \lambda \end{matrix}\right], \qquad (x, y) = E[z(\lambda) \mid 0 < \lambda < 1],$$
$$[x, y] = E[z(\lambda) \mid 0 \leq \lambda \leq 1], \qquad [x, y) = E[z(\lambda) \mid 0 \leq \lambda < 1],$$
$$(x, y] = E[z(\lambda) \mid 0 < \lambda \leq 1], \qquad (x, \vec{y}) = E[z(\lambda) \mid \lambda > 1],$$
$$(\overleftarrow{x}, y) = E[z(\lambda) \mid \lambda < 0],$$

where, of course,

$$\{x, y\} = \{y, x\}, \quad (x, y) = (y, x), \quad [x, y] = [y, x]$$
$$(x, \vec{y}) = (\overleftarrow{y}, x), \quad [x, y) = (y, x], \quad (x, y] = [y, x).$$

Let a and b be distinct points of Ω and let

$$c = (1-\alpha)a + \alpha b.$$

Lemma 1. *Given any* $\varepsilon > 0$, *we can find two positive numbers* $\delta_1 > 0$, $\delta_2 > 0$ *such that* $\{u, v\} \cap U(c; \varepsilon) \neq 0$ *for every* $u \in U(a; \delta_1)$ *and every* $v \in U(b; \delta_2)$. *Then it is said that* $U(a; \delta_1)$ *and* $U(b; \delta_2)$ *cross* $U(c; \varepsilon)$.

Proof. Set $\beta = Max.\left\{\left|\alpha - \dfrac{\varepsilon}{3\|a-b\|}\right|, \left|\alpha + \dfrac{\varepsilon}{3\|a-b\|}\right|\right\}$,

$$\gamma = Max.\left\{\left|1-\alpha+\frac{\varepsilon}{3\|a-b\|}\right|, \left|1-\alpha-\frac{\varepsilon}{3\|a-b\|}\right|\right\}.$$

For every u, v and λ such that $\|u-a\| < \delta_1 = \varepsilon/3\gamma$, $\|v-b\| < \delta_2 = \varepsilon/3\beta$, and $|\alpha-\lambda| < \varepsilon/3\|a-b\|$, it will be shown that $w = (1-\lambda)u + \lambda v$ belongs to $U(c; \varepsilon)$.

In fact, since $|\lambda| < \beta$ and $|1-\lambda| < \gamma$, we have

$$\|w-c\| \leq |1-\lambda| \, \|u-a\| + |\lambda| \, \|v-b\| + |\alpha-\lambda| \, \|a-b\|$$
$$< \gamma \cdot \frac{\varepsilon}{3\gamma} + \beta \cdot \frac{\varepsilon}{3\beta} + \frac{\varepsilon}{3\|a-b\|} \cdot \|a-b\| = \varepsilon.$$

Remark. If $0 < \varepsilon < 3 \cdot Min.\{\|a-c\|, \|b-c\|\}$, then λ satisfies $(\lambda-1)(\alpha-1) > 0$ and $\lambda\alpha > 0$. We say that $U(a; \delta_1)$ *and* $U(b; \delta_2)$ *cross separately* $U(c; \varepsilon)$.

Corollary 1. *If for any* ε, $0 < \varepsilon < 2 \cdot Min.\{\|a-c\|, \|b-c\|\}$, *we take any* v *and* λ *such that*

$$\|v-b\| < \delta_2 = \varepsilon/2\beta, \qquad |\alpha-\lambda| < \varepsilon/2\|a-b\|$$

where

$$\beta = Max.\left\{\left|\alpha - \frac{\varepsilon}{2\|a-b\|}\right|, \left|\alpha + \frac{\varepsilon}{2\|a-b\|}\right|\right\},$$

then $U(c\,;\,\varepsilon)$ *contains* $w=(1-\lambda)a+\lambda v$ *with* λ *satisfyiug both* $(\lambda-1)(\alpha-1)>0$ *and* $\lambda\alpha>0$.

It is said that a *and* $U(b\,;\,\delta_2)$ *cross separately* $U(c\,;\,\varepsilon)$.

Lemma 2. *Let* M *be a convex set,* $\overline{M}$ *the closure of* M. *If* $a\in M^i$ *and* $b\in\overline{M}$, *then* $[a,b)\subset M^i$ (Cf. [1]).

Proof. Suppose that $(a,b)\not\subset M^i$. Then (a,b) would contain $c\,\overline{\in}\,M^i$. Since $a\in M^i$, $U(a\,;\,\varepsilon)\subset M$ for some $\varepsilon<3\cdot Min.\{\|b-a\|,\ \|c-a\|\}$. By means of Lemma 1, we can find positive numbers ζ and η, such that $U(a\,;\,\varepsilon)$ is separately crossed by two neighborhoods: $U(b\,;\,\zeta)$ intersecting M, and $U(c\,;\,\eta)$ intersecting M'.[3] Letting $x\in U(b\,;\,\zeta)\frown M$, $z\in U(c\,;\,\eta)\frown M'$, and $y\in(x,\vec{z})\frown U(a\,;\,\varepsilon)\subset M$, we have $z\in[x,y]$, contrary to the convexity of M.

From this lemma we get at once:

Corollary 2. *Let* M *be a convex set,* M^* *its boundary, and,* M^e *its exterior. If* $a\in M^i$ *and* $r\in M^*$, *then* $(a,\vec{r})\subset M^e$.

Lemma 3. *If* $[a,b)\subset M^i$, *and* b *is a convex point of* M, *then there lies* $\delta>0$ *such that* $[a,z)\subset M^i$ *for any* $z\in U(b\,;\,\delta)\frown M$ (Cf. [2]).

Proof. To each $x\in[a,b]$, there corresponds $U(x\,;\,\varepsilon(x)/2)$ satisfying the following conditions:

$$U(x\,;\,\varepsilon(x))\subset M \quad \text{for} \quad x\in[a,b)\,,$$
$$U(x\,;\,\varepsilon(x))\frown M \ \text{ is convex for } \ x=b\,.$$

The system of $U(x\,;\,\varepsilon(x)/2)$ for all $x\in[a,b]$ covers $[a,b]$; however, since $[a,b]$ is compact, $[a,b]$ is covered by a finite system of $U_i=U(a_i\,;\,\varepsilon_i/2)$ $(i=1,\dots,n)$ where $\varepsilon_i=\varepsilon(a_i)$ and $a_i=(1-\alpha_i)a+\alpha_i b$ $(i=1,\dots,n)$ (Cf. [3]). Without loss of generality it may be assumed that

(i) $\alpha_1=0,\quad \alpha_n=1,\quad \alpha_i<\alpha_{i+1}\quad (i=1,\dots,n-1)$,
(ii) $U_i\not\subset U_j\quad (i\neq j)$, (iii) $U_i\subset M\quad (i=1,\dots,n-1)$,
(iv) $U_i\frown U_{i+1}\neq 0\quad (i=1,\dots,n-1)$.

As easily seen from them, we obtain

$$\frac{|\varepsilon_i-\varepsilon_j|}{2\,\|a-b\|}<|\alpha_i-\alpha_j| \quad \text{for} \quad i\neq j\,,$$

---

3) We denote by M' the complementary set of M

especially

$$\frac{|\varepsilon_i-\varepsilon_j|}{2\,\|a-b\|}<|\alpha_i-\alpha_j|<\frac{\varepsilon_i+\varepsilon_j}{2\,\|a-b\|} \quad \text{for} \quad i=j\pm 1\,,$$

and so the interval $[0, 1]$ is covered by the system of open sets:

$$V_i\equiv V_i(\alpha_i\,;\,\varepsilon_i/2\,\|a-b\|)=E[\lambda\,|\,|\lambda-\alpha_i|<\varepsilon_i/2\,\|a-b\|] \quad (i=1,\dots,n)\,.$$

Let

$$\delta=\underset{i=1,\dots,n}{Min.}\,\delta_i\,,$$

where

$$\delta_i=\varepsilon_i/2\beta_i\,,$$

$$\beta_i=Max.\left\{\left|\alpha_i-\frac{\varepsilon_i}{2\,\|a-b\|}\right|,\ \left|\alpha_i+\frac{\varepsilon_i}{2\,\|a-b\|}\right|\right\}, \quad (i=1,\dots,n)\,.$$

This δ will be what we desire here.

Setting $w(\lambda)\equiv(1-\lambda)a+\lambda z$ for any $z\in U(b\,;\,\delta)\frown M$, Corollary 1 shows that $w(\lambda)\in U(a_i\,;\,\varepsilon_i)$ for every $\lambda\in V_i$. Furthermore, if we take a real number ξ fulfilling

$$1-\frac{\varepsilon_n}{2\,\|a-b\|}<\xi<\alpha_{n-1}+\frac{\varepsilon_{n-1}}{2\,\|a-b\|}\,,$$

then for any $\lambda\in[0, \xi]$ there exists a positive integer k, i.e., one of $1, 2, \dots, n-1$ such that $\lambda\in V_k$, in other words, $w(\lambda)$ with any $\lambda\in[0, \xi]$ belongs to one of $U_i(i=1, 2, \dots, n-1)$; accordingly

$$[w(0)\,,\ w(\xi)]\subset M^i\,. \tag{1}$$

In particular, since $w(\xi)\in U_{n-1}\frown U_n\subset M$, $w(\xi)$ is an interior point of the convex set $U_n\frown M$.

Then by Lemma 2, it follows that

$$(w(\xi), z)\subset(U_n\frown M)^i\subset M^i\,. \tag{2}$$

Combining with (1) and (2), we have $[a, z)\subset M^i$. Thus this lemma has been proved.

3. The proof of the theorem.

Let a and b be any distinct points of M^i. By the assumption a and b are joined by an arc C in M i.e.,

$$C=E[z\,|\,z=f(\lambda),\ 0\leq\lambda\leq 1]\subset M$$

where $f(\lambda)$ represents a homeomorphic image of λ in M.

Now, let us define $L(\lambda)$ as following:

$$L(0)=\{a\}, \qquad L(\lambda)=(a, f(\lambda)) \quad \text{for } \lambda \neq 0.$$

Evidently $L(0)\subset M^i$. Since a is an interior point of M and $f(\lambda)$ is continuous, for any $\varepsilon>0$ there is β_0 such that $0<\beta_0<1$, $f(\beta)\in U(a;\varepsilon)\subset M^i$ for all β, $0<\beta<\beta_0$. Hence $L(\beta)\subset M^i$. Then we shall get $L(\lambda)\subset M^i$ for all $\lambda\in[0,1]$, whence the proof of this theorem is to be finished.

Supposing that it is not true, there is one at least λ yielding $L(\lambda)\not\subset M^i$. First setting

$$\mu=\inf_{L(\lambda)\not\subset M^i} \lambda, \tag{4}$$

where μ clearly lies in $[\beta_0, 1]$, we shall prove that $L(\mu)$ i.e.,

$$L(\mu)=(a, f(\mu))=[x(\nu)\mid x(\nu)=(1-\nu)a+\nu f(\mu), \quad 0<\nu<1]$$

contains one at least point of M^*. To do this it is sufficient to show that only $L(\mu)\not\subset M^i$ because really $x(\nu)\in M^i$ for at least every $\nu\in[0, \varepsilon/\|a-f(\mu)\|]$. Suppose $L(\mu)\subset M^i$. Since $f(\mu)$ is a convex point of M, there exists $\delta>0$ such that $(a,z)\subset M^i$ for any $z\in U(f(\mu);\delta)$ (by Lemma 3) and we take here particularly $z=f(\mu+\eta)$ such as shows below.
By continuity of $f(\lambda)$, we can select $\eta_0>0$ such that

$$f(\mu+\eta)\in U(f(\mu);\delta) \quad \text{for every } \eta\in(-\eta_0, \eta_0).$$

Therefore $L(\mu+\eta)\subset M^i$ for every $\eta\in(-\eta_0, \eta_0)$, contradicting to the assumption (4).

Let us denote by ν_0 the infinimum of all ν for which $x(\nu)\in M^*\cap L(\mu)$; obviously we have

$$\frac{\varepsilon}{\|a-f(\mu)\|}\leqq \nu_0<1, \quad x(\nu_0)\in M^*\cap L(\mu)$$

and $$x(\nu)\in M^i \quad \text{for every } \nu, \ 0<\nu<\nu_0.$$

M is convex at $x(\nu_0)$, i.e., $U(x(\nu_0);\zeta)\cap M$ is convex for a suitable ζ; and if $\nu_0-\zeta/\|a-f(\mu)\|<\nu<\nu_0$,

$$x(\nu)\in U(x(\nu_0);\zeta)\cap M^i=(U(x(\nu_0);\zeta)\cap M)^i.$$

On account of Corollary 2, it follows that

$$x(\xi)\in U(x(\nu_0);\zeta)\cap M^e \quad \text{for all } \xi, \ \nu_0<\xi<\nu_0+\zeta/\|a-f(\mu)\|,$$

that is, $$U(x(\xi_0);\gamma)\subset U(x(\nu_0);\zeta)\cap M^e \quad \text{for some } \gamma>0.$$

Hence a and $U(f(\mu);\sigma)$ cross $U(x(\xi_0);\gamma)$ if $\sigma>0$ is adequately chosen. On the other hand the continuity of $f(\mu)$ enables us to obtain $f(\mu-\delta)\in U(f(\mu);\sigma)$ for some $\delta>0$; so that a and $f(\mu-\delta)$ cross $U(x(\xi_0);\gamma)$, in other words, we have $L(\mu-\delta)\frown U(x(\xi_0);\gamma)\subset M^e$, i.e., $L(\mu-\delta)\not\subset M^i$, which arrives at the contradiction to $\mu=\inf_{L(\lambda)\not\subset M^i}\lambda$. Therefore $L(\lambda)\subset M^i$ for all $\lambda\in[0,1]$, especially, $L(1)=(a,b)\subset M^i$. The proof of the theorem has been completed.

We can easily give an example verifying that M is not convex under the same assumption as the above theorem. For example, let M be a set of points in the plane with cartesian coordinates $((x,y))$ satisfying

$$|y|\leqq 1 \quad \text{if} \quad \frac{1}{3}\leqq |x|\leqq 1\,,$$

$$|y|<1 \quad \text{if} \quad |x|<\frac{1}{3}\,.$$

Gakugei Faculty, Tokushima University.

Notes.

[1] By the way, it follows immediately from Lemma 2 that if M is convex M^i is convex. Moreover, it is likewise proved that if M is convex $\bar{M}$ is so.

[2] Lemma 3 holds even if we let a be, more generally, a convex point of M.

[3] Ω is regular and perfectly separable, because Ω is a separable metric space. Therefore Borel's covering theorem holds.

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

SOME PROPERTIES ON GEOMETRY IN COMPLEX SPACE (Part I)

By

Takaharu MARUYAMA

(Received Sept. 30, 1950)

Introduction.

If we consider a complex point in n-dimensional space, that is, whose coordinates are complex numbers, as a real point in 2n-dimensional space, we can construct a geometry in complex space. As a particular case of the above, we can represent the images of real and complex plane curves and straight lines as real surfaces and planes in 4-dimensional space. In the first place, we shall describe the important properties of these planes.

§1. Holomorphic planes.

Let us put complex variables z^1, z^2, and complex constants α_1, α_2, γ to x^1+iy^1, x^2+iy^2, and A_1+iB_1, A_2+iB_2, $C+iD$, respectively, then we obtain the equation of a straight line on a real plane, as

$$\alpha_1 z^1+\alpha_2 z^2+\gamma=0 \qquad (1)$$

If we put the real and imaginary part of the above equation to zero, we have the following equations:

$$\left.\begin{aligned} A_1x^1-B_1y^1+A_2x^2-B_2y^2+C=0, \\ B_1x^1+A_1y^1+B_2x^2+A_2y^2+D=0. \end{aligned}\right\} \qquad (2)$$

These equations define a plane of a special type in 4-dimensional space. We shall call these planes *holomorphic planes* and describe their remarkable properties.

I. *Two non-parallel holomorphic planes are not contained in a same hyperplane.*

Let the given two holomorphic planes be

$$\left.\begin{array}{r}A_1x^1-B_1y^1+A_2x^2-B_2y^2+C=0\\ B_1x^1+A_1y^1+B_2x^2+A_2y^2+D=0\end{array}\right\}\ \text{and}$$
$$\left.\begin{array}{r}A_1'x^1-B_1'y^1+A_2'x^2-B_2'y^2+C'=0\\ B_1'x^1+A_1'y^1+B_2'x^2+A_2'y^2+D'=0\end{array}\right\}$$

The necessary and sufficient condition that the above two holomorphic planes are contained in a hyperplane is

$$\begin{vmatrix} A_1 & -B_1 & A_2 & -B_2 \\ B_1 & A_1 & B_2 & A_2 \\ A_1' & -B_1' & A_2' & -B_2' \\ B_1' & A_1' & B_2' & A_2' \end{vmatrix} = 0 \tag{3}$$

But the equation (3) may be able to transform to the form

$$(A_1A_2'-B_1B_2'-A_2A_1'+B_2'B_1')^2+(A_1B_2'+B_1A_2'-B_2A_1'-A_2B_1')^2=0$$

It is impossible because these planes are not parallel.

II. *Transformations which transform a holomorphic plane to a holomorphic plane.*

We shall look for an affine transformation which transforms a holomorphic plane to a holomorphic plane.
Let us consider an affine transformation

$$\begin{aligned}\bar{x}^1&=a_{11}x^1+b_{11}y^1+a_{12}x^2+b_{12}y^2+k_1,\\ \bar{y}^1&=a_{21}x^1+b_{21}y^1+a_{22}x^2+b_{22}y^2+k_2,\\ \bar{x}^2&=a_{31}x^1+b_{31}y^1+a_{32}x^2+b_{32}y^2+k_3,\\ \bar{y}^2&=a_{41}x^1+b_{41}y^1+a_{42}x^2+b_{42}y^2+k_4.\end{aligned}$$

If we get the conditions that the above transformation may be transform a holomorphic plane to a holomorphic plane, we obtain

$$a_{12}=b_{22},\ a_{22}=-b_{12},\ a_{32}=b_{42},\ a_{42}=-b_{32},$$
$$a_{11}=b_{21},\ a_{21}=-b_{11},\ a_{31}=b_{41},\ a_{41}=-b_{31}.$$

Hence such an affine transformation is shown as follows:

$$\left.\begin{aligned}\bar{x}^1&=a_{11}x^1-b_{11}y^1+a_{12}x^2-b_{12}y^2+k_1,\\ \bar{y}^1&=b_{11}x^1+a_{11}y^1+b_{12}x^2+a_{12}y^2+k_2,\\ \bar{x}^2&=a_{31}x^1-b_{31}y^1+a_{32}x^2-b_{32}y^2+k_3,\\ \bar{y}^2&=b_{31}x^1+a_{31}y^1+b_{32}x^2+a_{32}y^2+k_4.\end{aligned}\right\} \tag{4}$$

We shall call this transformation a holomorphic transformation. It is easily shown that all of the holomorphic transformations form a transformation group, and that holomorphic planes are invariant under the holomolphic transformations. We shall describe some remarkable properties of holomorphic planes in the followings.

§2. Some remarkable properties on plane complex geometry.

I. To a real or complex point on the plane, there corresponds a real point in 4-dimensional space.

II. To a real or complex straight line on the plane, there corresponds a holomorphic plane in 4-dimensional space.

III. To a real point on the plane, there corresponds a point on the Real Plane, i. e. the locus of real points on the plane, in 4-dimensional space. The Real Plane is not a holomorphic plane.

IV. Two non-parallel holomorphic planes are not contained in a same hyperplane. Then two holomorphic planes have one and only one point in commom.

V. With the Real Plane a holomorphic plane, which represents a complex straight line, determines one and only one point, and that which represents a real straight line, determines a straight line. Then there is one and only one real point on a given complex straight line.

VI. The Real Plane and a complex point, i. e. a real point in 4-dimensional space, determine a hyperplane. The holomorphic planes which are contained in a same hyperplane are all parallel with one another. Then there is one and only one holomorphic plane which passes through a given point in the hyperplane. This holomorphic plane determines a straight line with the Real Plane. Then there is one and only one real straight line which passes through a given complex point.

§3. Isoclinic planes.

We have described in the Scientific Paper of Engineering, Tokushima University, Vol. 1, 2, No. 1, about the angles between two planes in 4-dimensional space. In this paper we shall describe briefly the abstract of the result.

Let the equations of the two given planes be

$A_i x^i = 0,\ B_i x^i = 0$ (I) and $A_i' x^i = 0,\ B_i' x^i = 0$ (II) $(i = 1, 2, 3, 4.)$

If we use the Gauss's notation $[AB]$ etc, such as $[AB] = \sum A_i B_i (i=1, 2, 3, 4.)$, we can put $[AA]=[BB]=[A'A']=[B'B']=1$, $[AB]=[A'B']=0$ without generality. The intersection of the plane (I) and the hyperplane which containes the plane (II), is a straight line. Then if we rotate the hyperplane about the plane (II), the straight line of the intersection generates an elliptic cone about the intersecting point of the planes. Then if we consider the conditions that the straight line generates a circular cone, we obtain the followings.

$$[A'B] = [AB'],\ [AA'] = -[BB'],\ \text{or}\ [A'B] = -[AB'],\ [AA'] = [BB'].$$

In the case of the above, the angles between the two given planes are determined uniquely, so we define such pairs of planes to *Isoclinic Planes*. We shall show some remarkable properties with respect to Isoclinic Planes.

I. *Any two holomorphic planes are usualy isoclinic mutually.*

Let the equations of the given holomorphic planes be

$$\begin{aligned} A_1x^1 - B_1y^1 + A_2x^2 - B_2y^2 + C &= 0, \\ B_1x^1 + A_1y^1 + B_2x^2 + A_2y^2 + D &= 0, \\ A_1'x^1 - B_1'y^1 + A_2'x^2 - B_2'y^2 + C' &= 0, \\ B_1'x^1 + A_1'y^1 + B_2'x^2 + A_2'y^2 + D' &= 0, \end{aligned}$$

then we have obviously $[AA']=[BB']$, $[A'B]=-[AB']$,

II. *An isoclinic plane to a given holomorphic plane is holomorphic.*

Let the equations of the given holomorphic plane be

$$\begin{aligned} A_1x^1 - B_1y^1 + A_2x^2 - B_2y^2 &= 0, \\ B_1x^1 + A_1y^1 + B_2x^2 + A_2y^2 &= 0, \end{aligned}$$

and that of the plane which is isoclinic to the former be

$$\begin{aligned} A_1'x^1 + A_2'y^1 + A_3'x^2 + A_4'y^2 &= 0, \\ B_1'x^1 + B_2'y^1 + B_3'x^2 + B_4'y^2 &= 0. \end{aligned}$$

Then if we apply the isoclinic properties, from the condition $[A'B]=[AB']$,

$[AA']=-[BB']$, and $[A'B]=-[AB']$, $[AA']=[BB']$, we get the equations

$$\left.\begin{aligned}(A_1'+B_2')A_1-(A_2'-B_1')B_1+(A_3'+B_4')A_2-(A_4'-B_3')B_2=0,\\(A_1'+B_2')B_1+(A_2'-B_1')A_1+(A_3'+B_4')B_2+(A_4'-B_3')A_2=0,\end{aligned}\right\}\text{(III) and}$$

$$\left.\begin{aligned}(A_1'-B_2')A_1-(A_2'+B_1')B_1+(A_3'-B_4')A_2-(A_4'+B_3')B_2=0,\\(A_1'-B_2')B_1+(A_2'+B_1')A_1+(A_3'-B_4')B_2+(A_4'+B_3')A_2=0.\end{aligned}\right\}\text{(IV)}$$

These show that the magnitudes in the parenthese satisfy the equation (I).

Then we put

$$\begin{aligned}x^1&=(A_1'+B_2'),\\y^1&=(A_2'-B_1'),\\x^2&=(A_3'+B_4'),\\y^2&=(A_4'-B_3'),\end{aligned}\qquad\text{or}\qquad\begin{aligned}x^1&=(A_1'-B_2'),\\y^1&=(A_2'+B_1'),\\x^2&=(A_3'-B_4'),\\y^2&=(A_4'+B_3').\end{aligned}$$

We see that x^i, y^i, $(i=1, 2.)$ satisfy the equation (I).

Applying the condition $[A'B']=0$, from the above we get the equations

$$\left.\begin{aligned}B_1'x^1+B_2'y^1+B_3'x^2+B_4'y^2=0,\\A_2'x^1-A_1'y^1+A_4'x^2-A_3'y^2=0,\end{aligned}\right\}\text{(V)}$$

$$\left.\begin{aligned}A_1'x^1+A_2'y^1+A_3'x^2+A_4'y^2=0,\\B_2'x^1-B_1'y^1+B_4'x^2-B_3'y^2=0.\end{aligned}\right\}\text{(VI)}$$

Then the determinant of the coefficients of the equation (I) and (V) or (VI) is not zero generally, for the existence of x^i, y^i satisfying these equations simultaneously, they must be zero respectively, we get the conditions.

$$\begin{aligned}A_1'+B_2'&=0,\\A_2'-B_1'&=0,\\A_3'+B_4'&=0,\\A_4'-B_3'&=0,\end{aligned}\qquad\text{or}\qquad\begin{aligned}A_1'-B_2'&=0,\\A_2'+B_1'&=0,\\A_3'-B_4'&=0,\\A_4'+B_3'&=0,\end{aligned}$$

It is shown that the plane (II) is a holomorphic plane.

The Faculty of Engineering, Tokushima University

Journal of Science of the Gakugei Faculty,
Tokushima University
Vol. 1, 1950

CHARACTERIZATION OF GROUPOIDS AND SEMILATTICES BY IDEALS IN A SEMIGROUP.

By Takayuki TAMURA

(Received Dec. 23, 1950)

§ 1. Introduction.

By a semigroup is meant a set E with an operation satisfying the conditions as following:

(1) To each pair of elements a and b of E, taken in this order, there corresponds a unique element $ab \in E$.

(2) The operation is associative, i. e., $(ab)c=a(bc)$.

In the present paper we shall not touch upon general theories of semigroups, but mainly discuss how a semilattice[1] and groupoids[2] are characterized by ideals or ideal systems in a semigroup.

§ 2. Ideals.

Definition. A non-empty subset I of a semigroup E is called a *right ideal* of E if $IE \subset I$[3]; a *left ideal* of E if $EI \subset I$; a *two-sided ideal* if $IE \subset I$ and $EI \subset I$; and "*universal*" is added if $=$ is taken in place of $\subset$.

Definition. Principal ideals are defined as following;

(1) I is called a *principal right ideal* if $aE=I$ for some $a \in I$,

(2) I a *principal left ideal* if $Eb=I$ for some $b \in I$,

(3) I a *principal two-sided ideal* if $aE=Eb=I$ for some $a, b \in I$.

And a (or b) is called a base of a principal right (left) ideal I, or I is said to be generated by a (or b); if a base need be assigned, we denote it by $R(a)$ or $L(b)$ according as it is right or left, and by $I(a)$ if we need not distinguish right from left.

E itself is a two-sided ideal. Ideals with the exception of E are to be called "proper" ones.

The following theorem is very easily shown and it holds even if "left" is

---

1) See § 5 in this paper.

2) See § 3.

3) We mean by IE the set of ax where $a \in I$, $x \in E$.

taken in place of " right ".

Theorem 1. *If $I_\alpha(\alpha \in M)$ are right ideals of a semigroup E, then so are also the union and the intersection of them, as far as the intersection is non-null.*

Consider the right (or left) ideal system $\mathfrak{I}^r$ (or $\mathfrak{I}^l$) i. e., the system composed of all right (left) ideals in E. Such a system is generally a semi-group—really a semilattice under the operation $I_\alpha \cup I_\beta$.

By the way, $\mathfrak{I}^r \frown \mathfrak{I}^l$ is the two-sided ideal system in E. The principal right (or left) ideal system $\mathfrak{P}^r$ (or $\mathfrak{P}^l$) can be similarly considered.

§ 3 Groupoids

Two kinds of groupoids [1] [2] are defined by restricting a semigroup.

Definition. A semigroup E is called a *right groupoid* if the following conditions are fulfilled. [3]

(1) E contains at least an element a such that there exists a left identity e depending on a, i. e., $ea=a$.

(2) Given any two elements a and c in E, we can find some $b \in E$ such that $ab=c$.

E is called a left groupoid if in (1) we take " right identity " instead of " left identity " and in (2) " $ba=c$ " instead of " $ab=c$ ".

The conditions (1) and (2) are replaced by (1′) and (2).

(1′) There is a left identity of E, i. e., an element e such that $ex=x$ for all $x \in E$.

In reality it can be shown that (1) and (2) imply (1′). Let $ea=a$ by (1); this e is nothing but a left identity of E. For, since any $x \in E$ is represented as $x=ay$ for some $y \in E$,

$$ex = e(ay) = (ea)y = ay = x.$$

It is clear that (1′) impiles (1). Thus (1) and (2) are equivalent to (1′) and (2).

It goes without saying that a group is a left groupoid as well as a right groupoid. Now, with respect to the relations between groupoids and their ideals, we have the following theorem.

Theorem 2. *In order that a semigroup E is a right groupoid (left groupoid), it is necessary and sufficient that E has no proper right (left) ideal, and has at least one left (right) principal ideal.*

Proof. Suppose that E is a right groupoid. Letting I be any right ideal of it, we have $xE \subset I \subset E$ for any $x \in I$; on the other hand $xE = E$ by the condition (2) of a right groupoid; hence $I = E$. Thus we see that there is no proper right ideal. The existence of a principal left ideal follows easily from (1). In fact Ea is a principal left ideal of E.

Next we shall prove the converse of this theorem. Since there exists no proper right ideal, we have, for any $x \in E$, $xE=E$, which immediately leads to (2). The condition (1), (consequently (1′)), follows[4] from the existence of a principal left ideal of E. The sufficiency has been completely proved.

The proof in the dual case is similar, needless to say.

From Theorem 2, we have without difficulty the following:

Theorem 3. *In order that a semigroup E is a group, it is necessary and sufficient that E has neither proper right nor proper left ideal.*

§ 4. Idempotency

Let x be an element of a semigroup E. An element x is called idempotent if $x = xx$, and E is called idempotent if all elements of E are idempotent. In this paragraph we shall assume E to be an idempotent semigroup. Then ideals, of course, are all universal.

Lemma 1. *If A is a right (left) ideal of E, and B is a right (left) ideal of A, then B is a right (left) ideal of E.*

Proof. We shall prove it only in the case of "right". BA is a right ideal of E. In fact, $(BA)E=B(AE) \subset BA$, and $BA \subset B$ since B is a right ideal of A; while $B \subset BA$ because E is idempotent and $B \subset A$; hence $B=BA$.

.**Lemma 2.** *The following three conditions are equivalent under the assumption of idempotency of E.*

(i) *$R(x) \in \mathfrak{J}^l$ and $L(x) \in \mathfrak{J}^r$ for every $x \in E$.*

(ii) *$R(x)=L(x)$ for every $x \in E$.*

(iii) *$\mathfrak{J}^r=\mathfrak{J}^l$.*

Proof. (i)→(ii) Since E is idempotent, $x \in R(x)$ and so $Ex \subset R(x)$ ($\because R(x) \in \mathfrak{J}^l$), therefore $L(x) \subset R(x)$. In the same way $L(x) \supset R(x)$, at last $L(x)=R(x)$.

(ii)→(iii) For $I \in \mathfrak{J}^r$ we have

4) Let $L(a)$ be a principal left ideal of E. By its definition there exists $e \in E$ such that $ea=a$.

$$I=\bigcup_{x \in I} R(x)=\bigcup_{x \in I} L(x) \in \mathfrak{I}^l,$$

hence $\mathfrak{I}^r \subset \mathfrak{I}^l$; similarly $\mathfrak{I}^r \supset \mathfrak{I}^l$, thus $\mathfrak{I}^r=\mathfrak{I}^l$.

(iii)→(i) It is evident.

Theorem 4. *If E is idempotent and $\mathfrak{I}^r=\mathfrak{I}^l$, then we have*

$$I(xy)=I(yx)=I(x) \cap I(y).$$

Proof. it is sufficient to show only the formula:

$$R(xy)=R(yx)=R(x) \cap R(y).$$

If $z \in R(x) \cap R(y)$, then $z=xa=yb$ for some $a, b \in E$. Utilizing this representation,

$$z=xa=x(xa)=x(yb)=(xy)b \in R(xy),$$

and so $R(x) \cap R(y) \subset R(xy)$.
Next, taking $z \in R(xy)$, we have

$$z=(xy)c=x(yc) \in R(x) \text{ for some } c \in E,$$

and $z=(xy)c=(yx')c$ (since $L(y)=R(y)$ by Lemma 2.)
$=y(x'c) \in R(y)$;
hence $R(yx) \subset R(x) \cap R(y)$, finally $R(x) \cap R(y)=R(xy)$. Interchanging x for y, $R(y) \cap R(x)=R(yx)$.

Further, respecting the relations between groupoids and idempotency, we have the following lemma and theorem.

Lemma 3. *In a right (left) groupoid G_r (G_l), $a \in G_r(G_l)$ is idempotent if and only if a is a left (right) identity.*

Proof. We shall prove it only in the case of G_r. Given any idempotent element a of the groupoid G_r, each $y \in G_r$ is formed as $y=ax$ for some $x \in G_r$; and we have

$$ay=a(ax)=(aa)x=ax=y \qquad \text{for all } y.$$

It follows that any $a \in G_r$ is a left identity of G_r. The converse is evident.

Theorem 5. *A group is idempotent if and only if it consists of only identity.*

Proof. Suppose that G is a group. Lemma 3 and the uniqueness of identity of a group show that all elements of G are equal. The converse is trivial.

§ 5. Semilattice.

A semilattice is defined as a commutative, idempotent semigroup [4].

Clearly this system coincides with a partly ordered set in which any pair of elements have a least upper bound or join — in fact, $y \geqq x$ meaning that $xy=y$ is a partial ordering, and xy is a join of x and y. For example, the right (or left) ideal system $\mathfrak{J}^r(\mathfrak{J}^l)$ is an above bounded semilattice under the operation $I_\alpha \cup I_\beta$, and $\mathfrak{J}^r$ (or $\mathfrak{J}^l$) includes E as the greatest element. We can show easily the following lemma and theorem.

Lemma 4. *In a semilattice E, $b \geqq a$ if and only if there exists snch $c \in E$ that $b=ac$.*

Theorem 6. *Let P be a principal ideal of the semilattice E. $P(b) \subset P(a)$ if and only if $a \leqq b$; accordingly $P(a)=P(b)$ implies $a=b$, that is, the mapping $x \leftrightarrow P(x)$ is one-to-one.*

By Theorem 6, we get readily the following theorem:

Theorem 7. *The principal ideal system $\mathfrak{P}$ of a semilattice E forms a semilattice under the operation $P(x) \cap P(y)$, and $\mathfrak{P}$ is isomorphic on E.*

Also we have:

Theorem 8. *The ideal system of a semilattice forms a lattice uuder the operations $\cup$ and $\cap$.*

From these theorems, we see that a minimal ideal implies the least ideal as far as semilattices are concerned; the system of all ideals in E containing x forms an above and below bounded lattice in which the principal ideal $P(x)$ is least.

maximal chain [5]. Let $\mathfrak{F}\{I_\omega \mid \omega \in \mathfrak{m}\}$ be a *maximal chain* in the ideal system $\mathfrak{J}^r$ or $\mathfrak{J}^l$ of the semigroup E, where $\mathfrak{m}$ denotes a totally ordered set having 0 as the least element, and $\sigma < \tau$ implies $I_\sigma \supset I_\tau$, and $I_0=E$. To any ideal I of E, there corresponds at least a maximal chain containing I; of course we must here assume the axiom of choice [6].

Setting

$$I' = \bigcap_{\substack{x \in I_\sigma \\ \sigma \in \mathfrak{m}}} I_\sigma \,,$$

I', being non-null, is an ideal of E by Theorem 1. Furthermore, $I' \subset I_\sigma$ for every I_σ ($\sigma \in \mathfrak{m}$) containing x; and $I_\sigma \supset I_\kappa$ for all I_σ and each I_κ ($\kappa \in \mathfrak{m}$) which does not contain x, so that $I' \supset I_\kappa$; I' is comparable with every $I_\omega \in \mathfrak{F}$, i. e., I' have to belong to $\mathfrak{F}$; and we can find $\lambda(x)$ in $\mathfrak{m}$ such that $I_{\lambda(x)}=I'$. This $I_{\lambda(x)}$ is nothing but the least of all ideals I_ω in $\mathfrak{F}$ which contain x.

Definition. The mapping $(x \to I_{\lambda(x)})$ of the semigroup E into $\mathfrak{F}$, i.e., on certain subset of $\mathfrak{F}$ is called a *natural mapping of* E *into* $\mathfrak{F}$, which is in general many to one.

Definition. Let $\mathfrak{F}$ be the one-sided ideal[5] system of the semigroup E. An ideal $A \in \mathfrak{F}$ is said to be *above isolated* if either of the following two holds:

(i) $A \subsetneqq C$ for no $C \in \mathfrak{F}$.

(ii) There exists $B \in \mathfrak{F}$ such that $A \subsetneqq C \subsetneqq B$ for no $C \in \mathfrak{F}$. If the sign $\supset$ is substituted for $\subset$, then A is said to be *below isolated*.

A one-sided ideal I of the semigroup E is below isolated if and only if I is the image of a suitable element $x \in E$ under the natural mapping of E into some $\mathfrak{F}$. By the way we have readily:

Lemma 5. *A principal ideal is below isolated.*

Now a semilattice will be characterized by the natural mapping and the maximal chain.

Theorem 9. *Let $\mathfrak{F}$ be the one-sided ideal system of a semigroup E, and $\mathfrak{F}$ any maximal chain in $\mathfrak{F}$. In order that E is a semilattice, it is necessary and sufficient that the following conditions are fulfilled.*

(1) *Every below isolated ideal is universal and two-sided.*

(2) *Every above isolated ideal is two-sided.*

(3) *The natural mapping of E into $\mathfrak{F}$ is one-to-one.*

Note. The condition (3) is equivalent to (3′).

(3′) For each $x \in E$, $I_{\lambda(x)} - I^*_\lambda = \{x\}$[6], where $I^*_\lambda = \bigcup_{\lambda(x) \succ \tau,\ \tau \in \mathfrak{M}} I_\tau$

Proof of necessity in Theorem 9.

Assume that E is a semilattice. The conditions (1) and (2) are clear because E is idempotent and commutative, and so we shall below prove (3′).

1° Letting $J = I_{\lambda(x)} - I^*_\lambda$, the definition of $\lambda(x)$ enables us to obtain that $x \in J$ and $\lambda(y) = \lambda(z)$ for any $y, z \in J$, in other words, $I_{\lambda(x)}$ is the least ideal belonging to $\mathfrak{F}$ which contains all $u \in J$.

2° We shall verify that if J was composed of at least two elements, it would contradict with the description 1°.

If there exists such an ideal K of $I_{\lambda(x)}$ that $J \frown K \neq 0$ and $J \not\subset K$, our

5) By one-sided ideal we meah either a right or left ideal.

6) $\{x\}$ represents the set composed of only a element x.

purpose is realized.

For, setting $L=K\cup I^*_\lambda$ which is an ideal of E because of Lemma 1, we have $L\supsetneqq I_\tau$ for every $\tau>\lambda$ since $J\frown K\neq 0$; and have $L\subsetneqq I_{\lambda(x)}$ since $J\not\subset K$; L must belong to $\mathfrak{F}$. The existence of such L in $\mathfrak{F}$ is in contradiction with the fact that $I_{\lambda(x)}$ is the least ideal in $\mathfrak{F}$ which contains $u\in J\frown K$.

3° In case that J is a subsemigroup.

We may assume without the loss of generality that J has a proper ideal of J. For, if J has none, J is a group by Theorem 3; and J contains only one element by Theorem 5 since J is idempotent.

Now, let M be a proper ideal of J and let $N=M\cup I^*_\lambda$. Then N is an ideal of $I_{\lambda(x)}$. In fact,

$$NI_{\lambda(x)}=(M\cup I^*_\lambda)(J\cup I^*_\lambda)=MJ\cup MI^*_\lambda\cup I^*_\lambda J\cup I^*_\lambda\subset M\cup I^*_\lambda=N,$$

moreover $J\frown N\neq 0$, $J\not\subseteq N$; the problem is reduced to 2°.

4° In case that J is not a subsemigroup.

There lie two (different) elements a, b in J such that $ab\,\overline{\in}\,J$, i. e., $ab\in I^*_\lambda$. Consider the two principal ideals $P(a)$ and $P(b)$ of E; we get $P(a)\frown P(b)=P(ab)$ by Theorem 4, but since $ab\in I^*_\lambda$ and I^*_λ is an ideal, $P(ab)\subset I^*_\lambda$; hence $b\,\overline{\in}\,P(a)$. Put $H=P(a)\cup I^*_\lambda$. Then $J\frown H\neq 0$, $J\not\subset H$; the problem is also reduced to 2°.

We have thus proved that $J=\{x\}$ in all cases; thus the proof of necessity has been completed.

proof of sufficiency.

1° *Proof of idempotency.*

Conversely assume that (1), (2), and (3) are satisfied. By (3′) we have $x\in I_{\lambda(x)}=I^*_\lambda\cup\{x\}$ where $x\,\overline{\in}\,I^*_\lambda$. The universality (1) of $I_{\lambda(x)}$ enables us to find some y and z in $I_{\lambda(x)}$ such that $x=yz$, but it is impossible to find neither y nor z in I^*_λ since I^*_λ is two-sided (by (2)). Hence $y=z=x$; we have $x=xx$ for all x, thereby the idempotency of E is established.

2° *preparation*

In order to prove commutativity, we shall prepare the following lemma:

Lemma 6. *Let $P(x)$ and $P(y)$ be one-sided principal ideals of E. If any $\mathfrak{F}$ satisfies (3), then $P(x)=P(y)$ implies $x=y$.*

Proof. The assumption (3) and Lemma 5 lead immediately to this lemma.

3° *Proof of commutativity*

It follows from (1) that $R(x)\in\mathfrak{J}^l$ and $L(x)\in\mathfrak{J}^r$. Since E is idempotent, $R(x)=L(x)$ for every $x\in E$ by Lemma 2; and $I(xy)=I(yx)$ by Theorem 4; therefore $xy=yx$ for every $x\in E$ and $y\in E$ (by Lemma 6.).

Thus the proof of Theorem 9 is completely finished.

Corollary. *In the semilattice E, a minimal (or least) ideal of E, if exists, consists of only one element.*

Proof. The proof of the corollary is established in that of Theorem 9 in which we may let I^*_λ be null.

Gakugei Faculty, Tokushima University.

Notes.

〔1〕 K. Masuda: Notes on groups (Japanese), Zenkoku Sizyo Sugaku Danwakai, v. 2, No. 11, pp. 338--341, 1948. He called our groupoid S-group.

〔2〕 K. Shoda: The general theory of Algebra (Japanese), Kyoritsusha, pp. 66—69, 1947.

〔3〕 If E is finite, the condition (1) is needless, that is, (1) follows from (2) and associative law. It is, however, doubtful for me, whether this holds in the case that E is infinite. If (1) can be omitted, Theorem 2 should be more simple.

〔4〕 G. Birkhoff: Lattice theory, revised edition, (Amer. Math. Soc. Colloq. Publ. 25) p. 18, 1948. "Semilattices" are due to Fr. Klein, Math. Zeits. 48 (1943), but I have not read it.

〔5〕 C is called a maximal chain if (i) C is a chain in $\mathfrak{J}$, and (ii) for any ideal $I\in\mathfrak{J}-C$, I is incomparable to any ideal belonging to C. With respect to chains, see G. Birkhoff: Lattice theory, Chapt. III, 1948.

〔6〕 G. Birkhoff: Lattice theory p. 42, 1948.

ABSTRACTS

An Example of a Non-normal Distribution Function with $\alpha_3=0,\ \alpha_4=3$

By Yoshikatsu WATANABE

Since Pearson's types of frequency functions are limited as solutions of $\frac{dy}{dx}=\frac{x-m_0}{a+bx+cx^2}$, the only function with $\alpha_3=0$, $\alpha_4=3$ is decided to be the normal distribution $y=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\bar{x})^2}{2\sigma^2}\right\}$. However as the world of statistics are broad, and there exist many functions such that $\frac{dy}{dx}=\frac{A+Bx+Cx^2+\cdots}{a+bx+cx^2+\cdots}$, and belong to no Pearson's type, the above conclusion is not reasonable. Though this fact is described in any book on statistics, I have never seen its actual example. I have found accidentally the following example:

$$y=f(x)=\frac{c}{(1+x^4)^2}$$

which gives $\alpha_3=0$ and $\alpha_4=3$. The integral computation being somewhat troublesome, it may serve as a good exercise on the theory of a complex variable.

Really putting

$$J_k=\oint\frac{cz^k}{(1+z^4)^2}dz\,,\quad k=0,1,2,3,4,$$

and making use of the method of the residuals we obtain the following results:

$$J_0=\frac{3\pi}{4\sqrt{2}}\,,\ J_1=J_3=0\,,\ J_2=J_4=\frac{\pi}{4\sqrt{2}}\,,$$

$$\therefore\quad c=1/J_0=\frac{4\sqrt{2}}{3\pi}\,,\ \text{and the mean}=0.$$

$$\sigma^2=cJ_2=J_2/J_0=\frac{1}{3}\,,\quad \alpha_3=0\,,$$

$$\alpha_4=cJ_4/\sigma^4=J_0J_4/J_2^2=3\,.$$

Gakugei Faculty,
Tokushima University.

On a Relation between the Radius of Analyticity and the Radius of Bound of the Power Series

By Isae SHIMODA

Let E, E' be two complex Banach spaces. An E' valued function $h_n(x)$ defined on E is an homogeneous polynomial of degree n. Then E' valued function $\sum_{n=0}^{\infty}h_n(x)$ is an power series defined on E. The radius of analyticity τ of $\sum_{n=0}^{\infty}h_n(x)$ is given by the following equality

$$\sup_{x\in k}\ \overline{\lim_{n\to\infty}}\ \sqrt[n]{\sup_{x\in G}\|h_n(x)\|}=\frac{1}{\tau}\,,$$

where k is a set of all compact set G extracted from the set $\|x\|=1$.

The radius of bound λ of $\sum_{n=0}^{\infty}h_n(x)$ is

$$\overline{\lim_{n\to\infty}}\ \sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=\frac{1}{\lambda}\,.$$

From this, we can easily see the following relation between λ and τ.

Theorem. Generally $\lambda\leq\tau$, but if the complex Banach spaces is compact, then $\lambda=\tau$.

In a complex n-dimensional spaces, $\lambda=\tau$.

Gakugei Faculty,
Tokushima University.

On Some Properties of Semiconvex Sets

By Takayuki TAMURA

We shall introduce the concept of semiconvexity and arrange some remarkable properties of semiconvex sets. Let x, y be points of the finite dimensional real Banach space Ω. By the notations $[x, y]$, $[\overleftarrow{x}, \overrightarrow{y}]$ we mean that

$[x, y] = E[(1-\lambda)x+\lambda y \mid 0 \leqq \lambda \leqq 1]$,

$[\overleftarrow{x}, \overrightarrow{y}] = E[(1-\lambda)x+\lambda y \mid \lambda \leqq 0,\ \lambda \geqq 1]$.

Definition. S is called a semiconvex set if one at least of the following conditions holds:

(i) $[x, y] \subset S$ (ii) $[\overleftarrow{x}, \overrightarrow{y}] \subset S$

for each pair of x, y belonging to S.

The above definition is equivalent to the following one.

Definition. If $[x, z] \cap S$ is connected for any $x \in S$ and any $z \in S'$ (the complementary set of S), then S is called a semiconvex set.

As easily seen from the definition, bounded semiconvex sets are convex; and if S is semiconvex S' is also so. Semiconvexity is invariant by the linear transformation $x \rightarrow \xi x + a$ where a is a definite point in Ω and ξ a constant number. Furthermore it is shown that the boundary S^* of the semiconvex set S coincides with that of the interior S^i of S.

Definition. A point x is called a direct point of M if x is a convex point[1] of M as well as M'; x an inner convex point of M if a convex point of M^i; x an inner non-convex point of M if not so; x an inner direct point of M if a convex point of M^i as well as M'^i.

In case that a semiconvex set S has no other than an inner direct point, its structure is very simple, that is, S^* consists of at most two hyperplanes.

In other cases, we have some remarkable results as following.

Lemma. Let o, a_1, a_2 be linearly independent three points of the semiconvex set S in which $dim\ S \geqq 2$. If $[o, a_1] \not\subset S$ and $[o, a_2] \not\subset S$, then every point of $S \cap R(o, a_1, a_2)$[2] is joined with one at least of o, a_1, a_2 by an arc contained in S.

Theorem. If there is one at least boundary point r of S such that

(i) r is an inner non-convex point of S,

(ii) r is an arcwise connected point[3] of S,

then S is arcwise connected.

Theorem. The semiconvex set consists of at most two components.[4]

Theorem. If A and B are components of an arcwise disconnected semiconvex set S, then A^i and B^i are both convex.

These theorems enable us to conclude that connectedness and arcwise connectedness are equivalent as far as semiconvex sets are concerned.

Furthermore we have:

Theorem. If a semiconvex set S is disconnected, then S' is connected.

Example. If we let

$f(x, y) = \alpha x^2 + 2\nu xy + \beta y^2 + 2\mu x + 2\lambda y + \gamma$

with real coefficients, and P be the totality of (x, y) in the plane satisfying $f(x, y) > 0$, P is obviously a semiconvex domain.

Gakugei Faculty,
Tokushima University.

Remark.

1) Cf. p. 25 in this Journal.

2) By $R(o, a_1, a_2)$ we mean the minimal linear subspace containing o, a_1, and a_2.

3) If for any $\varepsilon > 0$ $U(r; \varepsilon) \frown S$ is arcwise connected, r is called an arcwise connected point.

4) By a component is meant a maximal arcwise connected subset.

On a Machine Solving Algebraic Equations of Higher Order with Real Coefficients

By Yoshio HAYASHI

Abstracted from Scientific Papers of Faculty of Engineering, Tokushima Univ., (Japanese) Vol. 2, No. 1, 1950, pp. 61—63.

If one puts $z = re^{i\theta}$, as the solution of the following algebraic equation of n-th order with real coefficients:

$$(1) \quad a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

then z corresponds to a point on the Gauss-plane with pôlar coordinates (r, θ), while $z^2, z^3, \ldots, z^n$ correspond to points whose coordinates are respectively $(r^2, 2\theta)$, $(r^3, 3\theta)$, ..., $(r^n, n\theta)$.

Suppose one hangs weights proportional to the coefficients a_i at the points which correspond to z^i, then each $a_i z^i$ represents the moment of gravity of the weight a_i concerning the origin, and the equation (1) becomes the condition that these moments are in equilibrium, that is, if the weisht a_1 is put at the point (r, θ) and other weights are put respectively at their right positions, the system of weights keeps its level plane, while the wrong position of a_i will incline it.

So one can solve the equation (1) by the machine the structure of which is as follows; there are n-bars, the weights move radically keeping their distances from the rotation center of the bars in the ratio of $r : r^2 : \cdots : r^n$.

And this system of bars and weights inclines or in equilibrium according to the motions of bars and weights.

The position (r, θ) of weight a_1 shows the solution of the equation (1), when the system is in equilibrium. The mechanism of these actions is made of some pairs of circular plates and some cylinders. And the curves $y = x^i$ are carved on the surfaces of the cylinders.

The Faculty of Engineering,
Tokushima University.

MISCELLANEOUS NOTES

Tokushima Daigaku Sugaku Danwakai

We hold the meeting, "*Sugaku Danwakai*", generally once a month at the Mathematical Institute of Gakugei Faculty, Tokushima University. The addresses given in the meetings for the last year are as following.

The 1st Meeting
May 1, 1950

Yoshio Hayashi, On a machine solving algebraic equations of higher order.

Takayuki Tamura, On local convexity of sets.

Takashi Hirajima, An invariant integration in a compact topological group.

Takaharu Maruyama, The angles of intersection of planes in 4-dimensional space.

Isae Shimoda, Analytic functions in Banach spaces.

Yoshikatsu Watanabe, ω^2-distribution.

The 2nd Meeting
June 3, 1950

Seiichi Taga, On numerical calculations by the Punched card method.

Mamoru Matsuoka, On noted mathematicians of Tokushima-ken, seen from historical stand point.

The 3rd Meeting
July 18, 1950

Takeo Igarashi, The chronological table of mathematical history.

Takayuki Tamura, On semilattices.

The 4th Meeting
September 30. 1950

Takaharu Maruyama, Geometry in complex-spaces.

Isae Shimoda, A note on power series in abstract spaces (I).

Yoshikatsu Watanabe, A proof of Peters' formula.

Takayuki Tamura, On cubic matrices.

The 5th Meeting
November 30, 1950

Isae Shimoda, On power series in abstract spaces (II).

Takayuki Tamura, On ideals in universal semigroups (1).

The 6th Meeting
December 19, 1950

Yoshikatsu Watanabe, The generalization of Student's ratio (1).

Takaharu Maruyama, Geometry of path.

The 7th Meeting
January 25, 1951

Mamoru Matsuoka, Graphical meaning of imaginary points in some algebraic equations.

Takayuki Tamura, On ideals in universal semigroups (2).

Yoshikatsu Watanabe, The generalization of Student's ratio (2).

The 8th Meeting
February 22, 1951

Takashi Hirajima, On the fixed point theorem.

The 9th Meeting
March 15, 1951

Takaharu Maruyama, Geometry in Complex-spaces. (the relations with unitary spaces.)

Takayuki Tamura, Notes on semigroups.

The 10th Meeting
April 27, 1951

Yoshio Hayashi, The second fundamental quantities of regular surfaces.

Isae Shimoda, On a relation between the radius of analyticity and the radius of bound of power series.

The 11th Meeting
May 25, 1951

Seiichi Taga, A theorem in the theory of integral equations.

Takayuki Tamura, On a relation between finiteness and idempotency of semigroups.

徳島大學學藝學部
理科紀要
第 1 巻

昭和 26年 6 月25日 印刷
昭和 26年 7 月30日 発行

編輯兼発行者 徳島大学学藝学部
徳島市南常三島町

編輯責任者 渡 辺 義 勝

印刷者 小 林 積 造

印刷所 日本印刷出版株式会社
大阪市福島區亀甲町二丁目六二番地

PUBLISHED
BY GAKUGEI FACULTY
TOKUSHIMA UNIVERSITY
Tokushima, Japan

CONTENTS

ABSTRACTS