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CHARACTERIZATION OF GROUPOIDS AND SEMILATTICES BY IDEALS IN A SEMIGROUP.

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§ 1. Introduction.

By a semigroup is meant a set E with an operation satisfying the conditions as following:

(1) To each pair of elements a and b of E, taken in this order, there corresponds a unique element $ab \in E$.

(2) The operation is associative, i. e., $(ab)c=a(bc)$.

In the present paper we shall not touch upon general theories of semigroups, but mainly discuss how a semilattice[1] and groupoids[2] are characterized by ideals or ideal systems in a semigroup.

§ 2. Ideals.

Definition. A non-empty subset I of a semigroup E is called a *right ideal* of E if $IE \subset I$[3]; a *left ideal* of E if $EI \subset I$; a *two-sided ideal* if $IE \subset I$ and $EI \subset I$; and "*universal*" is added if $=$ is taken in place of $\subset$.

Definition. Principal ideals are defined as following;

(1) I is called a *principal right ideal* if $aE=I$ for some $a \in I$,

(2) I a *principal left ideal* if $Eb=I$ for some $b \in I$,

(3) I a *principal two-sided ideal* if $aE=Eb=I$ for some $a, b \in I$.

And a (or b) is called a base of a principal right (left) ideal I, or I is said to be generated by a (or b); if a base need be assigned, we denote it by $R(a)$ or $L(b)$ according as it is right or left, and by $I(a)$ if we need not distinguish right from left.

E itself is a two-sided ideal. Ideals with the exception of E are to be called "proper" ones.

The following theorem is very easily shown and it holds even if "left" is

1) See § 5 in this paper.

2) See § 3.

3) We mean by IE the set of ax where $a \in I$, $x \in E$.

taken in place of "right".

Theorem 1. *If $I_\alpha (\alpha \in M)$ are right ideals of a semigroup E, then so are also the union and the intersection of them, as far as the intersection is non-null.*

Consider the right (or left) ideal system $\mathfrak{J}^r$ (or $\mathfrak{J}^l$) i. e., the system composed of all right (left) ideals in E. Such a system is generally a semigroup—really a semilattice under the operation $I_\alpha \cup I_\beta$.

By the way, $\mathfrak{J}^r \cap \mathfrak{J}^l$ is the two-sided ideal system in E. The principal right (or left) ideal system $\mathfrak{P}^r$ (or $\mathfrak{P}^l$) can be similarly considered.

§ 3 Groupoids

Two kinds of groupoids [1] [2] are defined by restricting a semigroup.

Definition. A semigroup E is called a *right groupoid* if the following conditions are fulfilled. [3]

(1) E contains at least an element a such that there exists a left identity e depending on a, i. e., $ea=a$.

(2) Given any two elements a and c in E, we can find some $b \in E$ such that $ab=c$.

E is called a left groupoid if in (1) we take "right identity" instead of "left identity" and in (2) "$ba=c$" instead of "$ab=c$".

The conditions (1) and (2) are replaced by (1′) and (2).

(1′) There is a left identity of E, i. e., an element e such that $ex=x$ for all $x \in E$.

In reality it can be shown that (1) and (2) imply (1′). Let $ea=a$ by (1); this e is nothing but a left identity of E. For, since any $x \in E$ is represented as $x=ay$ for some $y \in E$,

$$ex = e(ay) = (ea)y = ay = x.$$

It is clear that (1′) impiles (1). Thus (1) and (2) are equivalent to (1′) and (2).

It goes without saying that a group is a left groupoid as well as a right groupoid. Now, with respect to the relations between groupoids and their ideals, we have the following theorem.

Theorem 2. *In order that a semigroup E is a right groupoid (left groupoid), it is necessary and sufficient that E has no proper right (left) ideal, and has at least one left (right) principal ideal.*

Proof. Suppose that E is a right groupoid. Letting I be any right ideal of it, we have $xE \subset I \subset E$ for any $x \in I$; on the other hand $xE = E$ by the condition (2) of a right groupoid; hence $I = E$. Thus we see that there is no proper right ideal. The existence of a principal left ideal follows easily from (1). In fact Ea is a principal left ideal of E.

Next we shall prove the converse of this theorem. Since there exists no proper right ideal, we have, for any $x \in E$, $xE=E$, which immediately leads to (2). The condition (1), (consequently (1′)), follows[4] from the existence of a principal left ideal of E. The sufficiency has been completely proved.

The proof in the dual case is similar, needless to say.

From Theorem 2, we have without difficulty the following:

Theorem 3. *In order that a semigroup E is a group, it is necessary and sufficient that E has neither proper right nor proper left ideal.*

§ 4. Idempotency

Let x be an element of a semigroup E. An element x is called idempotent if $x = xx$, and E is called idempotent if all elements of E are idempotent. In this paragraph we shall assume E to be an idempotent semigroup. Then ideals, of course, are all universal.

Lemma 1. *If A is a right (left) ideal of E, and B is a right (left) ideal of A, then B is a right (left) ideal of E.*

Proof. We shall prove it only in the case of "right". BA is a right ideal of E. In fact, $(BA)E=B(AE) \subset BA$, and $BA \subset B$ since B is a right ideal of A; while $B \subset BA$ because E is idempotent and $B \subset A$; hence $B=BA$.

.**Lemma 2.** *The following three conditions are equivalent under the assumption of idempotency of E.*

(i) $R(x) \in \mathfrak{J}^l$ *and* $L(x) \in \mathfrak{J}^r$ *for every* $x \in E$.

(ii) $R(x)=L(x)$ *for every* $x \in E$.

(iii) $\mathfrak{J}^r=\mathfrak{J}^l$.

Proof. (i)→(ii) Since E is idempotent, $x \in R(x)$ and so $Ex \subset R(x)$ ($\because R(x) \in \mathfrak{J}^l$), therefore $L(x) \subset R(x)$. In the same way $L(x) \supset R(x)$, at last $L(x)=R(x)$.

(ii)→(iii) For $I \in \mathfrak{J}^r$ we have

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4) Let $L(a)$ be a principal left ideal of E. By its definition there exists $e \in E$ such that $ea=a$.

$$I=\bigcup_{x \in I} R(x)=\bigcup_{x \in I} L(x) \in \mathfrak{J}^l,$$

hence $\mathfrak{J}^r \subset \mathfrak{J}^l$; similarly $\mathfrak{J}^r \supset \mathfrak{J}^l$, thus $\mathfrak{J}^r = \mathfrak{J}^l$.

(iii)→(i) It is evident.

Theorem 4. *If E is idempotent and $\mathfrak{J}^r = \mathfrak{J}^l$, then we have*

$$I(xy) = I(yx) = I(x) \frown I(y).$$

Proof. it is sufficient to show only the formula:

$$R(xy) = R(yx) = R(x) \frown R(y).$$

If $z \in R(x) \frown R(y)$, then $z=xa=yb$ for some $a, b \in E$. Utilizing this representation,

$$z = xa = x(xa) = x(yb) = (xy)b \in R(xy),$$

and so $R(x) \frown R(y) \subset R(xy)$.
Next, taking $z \in R(xy)$, we have

$$z=(xy)c=x(yc) \in R(x) \text{ for some } c \in E,$$

and $z=(xy)c=(yx')c$ (since $L(y)=R(y)$ by Lemma 2.)
$=y(x'c) \in R(y)$;
hence $R(yx) \subset R(x) \frown R(y)$, finally $R(x) \frown R(y)=R(xy)$. Interchanging x for y, $R(y) \frown R(x)=R(yx)$.

Further, respecting the relations between groupoids and idempotency, we have the following lemma and theorem.

Lemma 3. *In a right (left) groupoid G_r (G_l), $a \in G_r(G_l)$ is idempotent if and only if a is a left (right) identity.*

Proof. We shall prove it only in the case of G_r. Given any idempotent element a of the groupoid G_r, each $y \in G_r$ is formed as $y=ax$ for some $x \in G_r$; and we have

$$ay = a(ax) = (aa)x = ax = y \qquad \text{for all } y.$$

It follows that any $a \in G_r$ is a left identity of G_r. The converse is evident.

Theorem 5. *A group is idempotent if and only if it consists of only identity.*

Proof. Suppose that G is a group. Lemma 3 and the uniqueness of identity of a group show that all elements of G are equal. The converse is trivial.

§ 5. Semilattice.

A semilattice is defined as a commutative, idempotent semigroup [4].

Clearly this system coincides with a partly ordered set in which any pair of elements have a least upper bound or join — in fact, $y \geqq x$ meaning that $xy=y$ is a partial ordering, and xy is a join of x and y. For example, the right (or left) ideal system $\mathfrak{J}^r(\mathfrak{J}^l)$ is an above bounded semilattice under the operation $I_\alpha \cup I_\beta$, and $\mathfrak{J}^r$ (or $\mathfrak{J}^l$) includes E as the greatest element. We can show easily the following lemma and theorem.

Lemma 4. *In a semilattice E, $b \geqq a$ if and only if there exists snch $c \in E$ that $b=ac$.*

Theorem 6. *Let P be a principal ideal of the semilattice E. $P(b) \subset P(a)$ if and only if $a \leqq b$; accordingly $P(a)=P(b)$ implies $a=b$, that is, the mapping $x \leftrightarrow P(x)$ is one-to-one.*

By Theorem 6, we get readily the following theorem:

Theorem 7. *The principal ideal system $\mathfrak{P}$ of a semilattice E forms a semilattice under the operation $P(x) \frown P(y)$, and $\mathfrak{P}$ is isomorphic on E.*

Also we have:

Theorem 8. *The ideal system of a semilattice forms a lattice uuder the operations $\cup$ and $\frown$.*

From these theorems, we see that a minimal ideal implies the least ideal as far as semilattices are concerned; the system of all ideals in E containing x forms an above and below bounded lattice in which the principal ideal $P(x)$ is least.

maximal chain [5]. Let $\mathfrak{F}\{I_\omega \mid \omega \in \mathfrak{m}\}$ be a *maximal chain* in the ideal system $\mathfrak{J}^r$ or $\mathfrak{J}^l$ of the semigroup E, where $\mathfrak{m}$ denotes a totally ordered set having 0 as the least element, and $\sigma < \tau$ implies $I_\sigma \supset I_\tau$, and $I_0=E$. To any ideal I of E, there corresponds at least a maximal chain containing I; of course we must here assume the axiom of choice [6].

Setting

$$I' = \bigcap_{\substack{x \in I_\sigma \\ \sigma \in \mathfrak{m}}} I_\sigma\,,$$

I', being non-null, is an ideal of E by Theorem 1. Furthermore, $I' \subset I_\sigma$ for every I_σ $(\sigma \in \mathfrak{m})$ containing x; and $I_\sigma \supset I_\kappa$ for all I_σ and each I_κ $(\kappa \in \mathfrak{m})$ which does not contain x, so that $I' \supset I_\kappa$; I' is comparable with every $I_\omega \in \mathfrak{F}$, i. e., I' have to belong to $\mathfrak{F}$; and we can find $\lambda(x)$ in $\mathfrak{m}$ such that $I_{\lambda(x)}=I'$. This $I_{\lambda(x)}$ is nothing but the least of all ideals I_ω in $\mathfrak{F}$ which contain x.

Definition. The mapping $(x \to I_{\lambda(x)})$ of the semigroup E into $\mathfrak{J}$, i. e., on certain subset of $\mathfrak{J}$ is called a *natural mapping of E into* $\mathfrak{J}$, which is in general many to one.

Definition. Let $\mathfrak{J}$ be the one-sided ideal[5] system of the semigroup E. An ideal $A \in \mathfrak{J}$ is said to be *above isolated* if either of the following two holds:

(i) $A \subsetneqq C$ for no $C \in \mathfrak{J}$.

(ii) There exists $B \in \mathfrak{J}$ such that $A \subsetneqq C \subsetneqq B$ for no $C \in \mathfrak{J}$. If the sign $\supset$ is substituted for $\subset$, then A is said to be *below isolated*.

A one-sided ideal I of the semigroup E is below isolated if and only if I is the image of a suitable element $x \in E$ under the natural mapping of E into some $\mathfrak{J}$. By the way we have readily:

Lemma 5. *A principal ideal is below isolated.*

Now a semilattice will be characterized by the natural mapping and the maximal chain.

Theorem 9. *Let $\mathfrak{J}$ be the one-sided ideal system of a semigroup E, and $\mathfrak{J}$ any maximal chain in $\mathfrak{J}$. In order that E is a semilattice, it is necessary and sufficient that the following conditions are fulfilled.*

(1) *Every below isolated ideal is universal and two-sided.*

(2) *Every above isolated ideal is two-sided.*

(3) *The natural mapping of E into $\mathfrak{J}$ is one-to-one.*

Note. The condition (3) is equivalent to (3′).

(3′) For each $x \in E$, $I_{\lambda(x)} - I^*_\lambda = \{x\}$[6], where $I^*_\lambda = \bigcup_{\lambda(x) \prec \tau,\ \tau \in \mathfrak{M}} I_\tau$

Proof of necessity in Theorem 9.

Assume that E is a semilattice. The conditions (1) and (2) are clear because E is idempotent and commutative, and so we shall below prove (3′).

1° Letting $J = I_{\lambda(x)} - I^*_\lambda$, the definition of $\lambda(x)$ enables us to obtain that $x \in J$ and $\lambda(y) = \lambda(z)$ for any $y, z \in J$, in other words, $I_{\lambda(x)}$ is the least ideal belonging to $\mathfrak{J}$ which contains all $u \in J$.

2° We shall verify that if J was composed of at least two elements, it would contradict with the description 1°.

If there exists such an ideal K of $I_{\lambda(x)}$ that $J \cap K \neq 0$ and $J \not\subset K$, our

5) By one-sided ideal we meah either a right or left ideal.

6) $\{x\}$ represents the set composed of only a element x.

purpose is realized.

For, setting $L=K\cup I_\lambda^*$ which is an ideal of E because of Lemma 1, we have $L\supsetneqq I_\tau$ for every $\tau>\lambda$ since $J\frown K\neq 0$; and have $L\subsetneqq I_{\lambda(x)}$ since $J\not\subset K$; L must belong to $\mathfrak{F}$. The existence of such L in $\mathfrak{F}$ is in contradiction with the fact that $I_{\lambda(x)}$ is the least ideal in $\mathfrak{F}$ which contains $u\in J\frown K$.

3° In case that J is a subsemigroup.

We may assume without the loss of generality that J has a proper ideal of J. For, if J has none, J is a group by Theorem 3; and J contains only one element by Theorem 5 since J is idempotent.

Now, let M be a proper ideal of J and let $N=M\cup I_\lambda^*$. Then N is an ideal of $I_{\lambda(x)}$. In fact,

$$NI_{\lambda(x)}=(M\cup I_\lambda^*)(J\cup I_\lambda^*)=MJ\cup MI_\lambda^*\cup I_\lambda^*J\cup I_\lambda^*\subset M\cup I_\lambda^*=N,$$

moreover $J\frown N\neq 0$, $J\not\subseteq N$; the problem is reduced to 2°.

4° In case that J is not a subsemigroup.

There lie two (different) elements a, b in J such that $ab\,\overline{\in}\,J$, i.e., $ab\in I_\lambda^*$. Consider the two principal ideals $P(a)$ and $P(b)$ of E; we get $P(a)\frown P(b)=P(ab)$ by Theorem 4, but since $ab\in I_\lambda^*$ and I_λ^* is an ideal, $P(ab)\subset I_\lambda^*$; hence $b\,\overline{\in}\,P(a)$. Put $H=P(a)\cup I_\lambda^*$. Then $J\frown H\neq 0$, $J\not\subset H$; the problem is also reduced to 2°.

We have thus proved that $J=\{x\}$ in all cases; thus the proof of necessity has been completed.

proof of sufficiency.

1° *Proof of idempotency.*

Conversely assume that (1), (2), and (3) are satisfied. By (3′) we have $x\in I_{\lambda(x)}=I_\lambda^*\cup\{x\}$ where $x\,\overline{\in}\,I_\lambda^*$. The universality (1) of $I_{\lambda(x)}$ enables us to find some y and z in $I_{\lambda(x)}$ such that $x=yz$, but it is impossible to find neither y nor z in I_λ^* since I_λ^* is two-sided (by (2)). Hence $y=z=x$; we have $x=xx$ for all x, thereby the idempotency of E is established.

2° *preparation*

In order to prove commutativity, we shall prepare the following lemma:

Lemma 6. *Let $P(x)$ and $P(y)$ be one-sided principal ideals of E. If any $\mathfrak{F}$ satisfies (3), then $P(x)=P(y)$ implies $x=y$.*

Proof. The assumption (3) and Lemma 5 lead immediately to this lemma.

3° *Proof of commutativity*

It follows from (1) that $R(x) \in \mathfrak{J}^l$ and $L(x) \in \mathfrak{J}^r$. Since E is idempotent, $R(x)=L(x)$ for every $x \in E$ by Lemma 2; and $I(xy)=I(yx)$ by Theorem 4; therefore $xy=yx$ for every $x \in E$ and $y \in E$ (by Lemma 6.).

Thus the proof of Theorem 9 is completely finished.

Corollary. *In the semilattice E, a minimal (or least) ideal of E, if exists, consists of only one element.*

Proof. The proof of the corollary is established in that of Theorem 9 in which we may let I_λ^* be null.

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Notes.

〔1〕 K. Masuda: Notes on groups (Japanese), Zenkoku Sizyo Sugaku Danwakai, v. 2, No. 11, pp. 338--341, 1948. He called our groupoid S-group.

〔2〕 K. Shoda: The general theory of Algebra (Japanese), Kyoritsusha, pp. 66—69, 1947.

〔3〕 If E is finite, the condition (1) is needless, that is, (1) follows from (2) and associative law. It is, however, doubtful for me, whether this holds in the case that E is infinite. If (1) can be omitted, Theorem 2 should be more simple.

〔4〕 G. Birkhoff: Lattice theory, revised edition, (Amer. Math. Soc. Colloq. Publ. 25) p. 18, 1948. "Semilattices" are due to Fr. Klein, Math. Zeits. 48 (1943), but I have not read it.

〔5〕 C is called a maximal chain if (i) C is a chain in $\mathfrak{J}$, and (ii) for any ideal $I \in \mathfrak{J}-C$, I is incomparable to any ideal belonging to C. With respect to chains, see G. Birkhoff: Lattice theory, Chapt. III, 1948.

〔6〕 G. Birkhoff: Lattice theory p. 42, 1948.