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SOME PROPERTIES ON GEOMETRY IN COMPLEX SPACE (Part I)

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Introduction.

If we consider a complex point in n-dimensional space, that is, whose coordinates are complex numbers, as a real point in 2n-dimensional space, we can construct a geometry in complex space. As a particular case of the above, we can represent the images of real and complex plane curves and straight lines as real surfaces and planes in 4-dimensional space. In the first place, we shall describe the important properties of these planes.

§1. Holomorphic planes.

Let us put complex variables z^1, z^2, and complex constants α_1, α_2, γ to x^1+iy^1, x^2+iy^2, and A_1+iB_1, A_2+iB_2, $C+iD$, respectively, then we obtain the equation of a straight line on a real plane, as

$$\alpha_1 z^1+\alpha_2 z^2+\gamma=0 \tag{1}$$

If we put the real and imaginary part of the above equation to zero, we have the following equations:

$$\left.\begin{aligned} A_1x^1-B_1y^1+A_2x^2-B_2y^2+C=0,\\ B_1x^1+A_1y^1+B_2x^2+A_2y^2+D=0. \end{aligned}\right\} \tag{2}$$

These equations define a plane of a special type in 4-dimensional space. We shall call these planes *holomorphic planes* and describe their remarkable properties.

I. *Two non-parallel holomorphic planes are not contained in a same hyperplane.*

Let the given two holomorphic planes be

$$\left.\begin{aligned} A_1x^1-B_1y^1+A_2x^2-B_2y^2+C=0 \\ B_1x^1+A_1y^1+B_2x^2+A_2y^2+D=0 \end{aligned}\right\} \text{ and}$$

$$\left.\begin{aligned} A_1'x^1-B_1'y^1+A_2'x^2-B_2'y^2+C'=0 \\ B_1'x^1+A_1'y^1+B_2'x^2+A_2'y^2+D'=0 \end{aligned}\right\}$$

The necessary and sufficient condition that the above two holomorphic planes are contained in a hyperplane is

$$\begin{vmatrix} A_1 & -B_1 & A_2 & -B_2 \\ B_1 & A_1 & B_2 & A_2 \\ A_1' & -B_1' & A_2' & -B_2' \\ B_1' & A_1' & B_2' & A_2' \end{vmatrix} = 0 \tag{3}$$

But the equation (3) may be able to transform to the form

$$(A_1A_2'-B_1B_2'-A_2A_1'+B_2'B_1')^2+(A_1B_2'+B_1A_2'-B_2A_1'-A_2B_1')^2=0$$

It is impossible because these planes are not parallel.

II. *Transformations which transform a holomorphic plane to a holomorphic plane.*

We shall look for an affine transformation which transforms a holomorphic plane to a holomorphic plane.
Let us consider an affine transformation

$$\begin{aligned} \bar{x}^1 &= a_{11}x^1+b_{11}y^1+a_{12}x^2+b_{12}y^2+k_1, \\ \bar{y}^1 &= a_{21}x^1+b_{21}y^1+a_{22}x^2+b_{22}y^2+k_2, \\ \bar{x}^2 &= a_{31}x^1+b_{31}y^1+a_{32}x^2+b_{32}y^2+k_3, \\ \bar{y}^2 &= a_{41}x^1+b_{41}y^1+a_{42}x^2+b_{42}y^2+k_4. \end{aligned}$$

If we get the conditions that the above transformation may be transform a holomorphic plane to a holomorphic plane, we obtain

$$a_{12}=b_{22},\ a_{22}=-b_{12},\ a_{32}=b_{42},\ a_{42}=-b_{32},$$
$$a_{11}=b_{21},\ a_{21}=-b_{11},\ a_{31}=b_{41},\ a_{41}=-b_{31}.$$

Hence such an affine transformation is shown as follows:

$$\left.\begin{aligned} \bar{x}^1 &= a_{11}x^1-b_{11}y^1+a_{12}x^2-b_{12}y^2+k_1, \\ \bar{y}^1 &= b_{11}x^1+a_{11}y^1+b_{12}x^2+a_{12}y^2+k_2, \\ \bar{x}^2 &= a_{31}x^1-b_{31}y^1+a_{32}x^2-b_{32}y^2+k_3, \\ \bar{y}^2 &= b_{31}x^1+a_{31}y^1+b_{32}x^2+a_{32}y^2+k_4. \end{aligned}\right\} \tag{4}$$

We shall call this transformation a holomorphic transformation. It is easily shown that all of the holomorphic transformations form a transformation group, and that holomorphic planes are invariant under the holomolphic transformations. We shall describe some remarkable properties of holomorphic planes in the followings.

§2. Some remarkable properties on plane complex geometry.

I. To a real or complex point on the plane, there corresponds a real point in 4-dimensional space.

II. To a real or complex straight line on the plane, there corresponds a holomorphic plane in 4-dimensional space.

III. To a real point on the plane, there corresponds a point on the Real Plane, i. e. the locus of real points on the plane, in 4-dimensional space. The Real Plane is not a holomorphic plane.

IV. Two non-parallel holomorphic planes are not contained in a same hyperplane. Then two holomorphic planes have one and only one point in commom.

V. With the Real Plane a holomorphic plane, which represents a complex straight line, determines one and only one point, and that which represents a real straight line, determines a straight line. Then there is one and only one real point on a given complex straight line.

VI. The Real Plane and a complex point, i. e. a real point in 4-dimensional space, determine a hyperplane. The holomorphic planes which are contained in a same hyperplane are all parallel with one another. Then there is one and only one holomorphic plane which passes through a given point in the hyperplane. This holomorphic plane determines a straight line with the Real Plane. Then there is one and only one real straight line which passes through a given complex point.

§3. Isoclinic planes.

We have described in the Scientific Paper of Engineering, Tokushima University, Vol. 1, 2, No. 1, about the angles between two planes in 4-dimensional space. In this paper we shall describe briefly the abstract of the result.

Let the equations of the two given planes be

$$A_i x^i = 0,\ B_i x^i = 0\ \text{(I) and}\ A_i' x^i = 0,\ B_i' x^i = 0\ \text{(II)} \quad (i = 1, 2, 3, 4.)$$

If we use the Gauss's notation $[AB]$ etc, such as $[AB] = \sum A_i B_i (i=1, 2, 3, 4.)$, we can put $[AA]=[BB]=[A'A']=[B'B']=1$, $[AB]=[A'B']=0$ without generality. The intersection of the plane (I) and the hyperplane which containes the plane (II), is a straight line. Then if we rotate the hyperplane about the plane (II), the straight line of the intersection generates an elliptic cone about the intersecting point of the planes. Then if we consider the conditions that the straight line generates a circular cone, we obtain the followings.

$$[A'B] = [AB'],\ [AA'] = -[BB'],\ \text{or}\ [A'B] = -[AB'],\ [AA'] = [BB'].$$

In the case of the above, the angles between the two given planes are determined uniquely, so we define such pairs of planes to *Isoclinic Planes*. We shall show some remarkable properties with respect to Isoclinic Planes.

I. *Any two holomorphic planes are usualy isoclinic mutually.*

Let the equations of the given holomorphic planes be

$$\begin{aligned} A_1 x^1 - B_1 y^1 + A_2 x^2 - B_2 y^2 + C &= 0, \\ B_1 x^1 + A_1 y^1 + B_2 x^2 + A_2 y^2 + D &= 0, \\ A_1' x^1 - B_1' y^1 + A_2' x^2 - B_2' y^2 + C' &= 0, \\ B_1' x^1 + A_1' y^1 + B_2' x^2 + A_2' y^2 + D' &= 0, \end{aligned}$$

then we have obviously $[AA'] = [BB']$, $[A'B] = -[AB']$,

II. *An isoclinic plane to a given holomorphic plane is holomorphic.*

Let the equations of the given holomorphic plane be

$$\begin{aligned} A_1 x^1 - B_1 y^1 + A_2 x^2 - B_2 y^2 &= 0, \\ B_1 x^1 + A_1 y^1 + B_2 x^2 + A_2 y^2 &= 0, \end{aligned}$$

and that of the plane which is isoclinic to the former be

$$\begin{aligned} A_1' x^1 + A_2' y^1 + A_3' x^2 + A_4' y^2 &= 0, \\ B_1' x^1 + B_2' y^1 + B_3' x^2 + B_4' y^2 &= 0. \end{aligned}$$

Then if we apply the isoclinic properties, from the condition $[A'B] = [AB']$,

$[AA']=-[BB']$, and $[A'B]=-[AB']$, $[AA']=[BB']$, we get the equations

$$\left.\begin{aligned}(A_1'+B_2')A_1-(A_2'-B_1')B_1+(A_3'+B_4')A_2-(A_4'-B_3')B_2=0,\\(A_1'+B_2')B_1+(A_2'-B_1')A_1+(A_3'+B_4')B_2+(A_4'-B_3')A_2=0,\end{aligned}\right\}\text{(III) and}$$

$$\left.\begin{aligned}(A_1'-B_2')A_1-(A_2'+B_1')B_1+(A_3'-B_4')A_2-(A_4'+B_3')B_2=0,\\(A_1'-B_2')B_1+(A_2'+B_1')A_1+(A_3'-B_4')B_2+(A_4'+B_3')A_2=0.\end{aligned}\right\}\text{(IV)}$$

These show that the magnitudes in the parenthese satisfy the equation (I).

Then we put

$$\begin{aligned}x^1&=(A_1'+B_2'),\\y^1&=(A_2'-B_1'),\\x^2&=(A_3'+B_4'),\\y^2&=(A_4'-B_3'),\end{aligned}\qquad\text{or}\qquad\begin{aligned}x^1&=(A_1'-B_2'),\\y^1&=(A_2'+B_1'),\\x^2&=(A_3'-B_4'),\\y^2&=(A_4'+B_3').\end{aligned}$$

We see that x^i, y^i, $(i=1, 2.)$ satisfy the equation (I).

Applying the condition $[A'B']=0$, from the above we get the equations

$$\left.\begin{aligned}B_1'x^1+B_2'y^1+B_3'x^2+B_4'y^2=0,\\A_2'x^1-A_1'y^1+A_4'x^2-A_3'y^2=0,\end{aligned}\right\}\text{(V)}$$

$$\left.\begin{aligned}A_1'x^1+A_2'y^1+A_3'x^2+A_4'y^2=0,\\B_2'x^1-B_1'y^1+B_4'x^2-B_3'y^2=0.\end{aligned}\right\}\text{(VI)}$$

Then the determinant of the coefficients of the equation (I) and (V) or (VI) is not zero generally, for the existence of x^i, y^i satisfying these equations simultaneously, they must be zero respectively, we get the conditions.

$$\begin{aligned}A_1'+B_2'&=0,\\A_2'-B_1'&=0,\\A_3'+B_4'&=0,\\A_4'-B_3'&=0,\end{aligned}\qquad\text{or}\qquad\begin{aligned}A_1'-B_2'&=0,\\A_2'+B_1'&=0,\\A_3'-B_4'&=0,\\A_4'+B_3'&=0,\end{aligned}$$

It is shown that the plane (II) is a holomorphic plane.

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