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ON A RELATION BETWEEN LOCAL CONVEXITY AND ENTIRE CONVEXITY

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1. Introduction.

The space considered here is a separable real-Banach space, written Ω. Let us denote points of Ω by $a, b, x, \ldots$ etc.; sets by $M, E, \ldots$ etc.; and real numbers by $\alpha, \lambda, \ldots$ etc.. If there exists $\delta > 0$ such that $U(x\,;\,\varepsilon) \frown M$,[1] as far as non-null, is convex for any positive $\varepsilon \leq \delta$, the point x is called a *convex point* of M, or M is said to be (locally) convex at x. As M is always convex at its interior points, the concept of local convexity at x is of special significance in the case x is a boundary point of M, and so the convex point of M implies an interior point or boundary point of M. When M is locally convex everywhere at the boundary, M is said to be locally convex.

Clearly, although convexity (in large) implies local convexity, the converse is not true. In this paper we impose on the local convex set M the condition of its arcwise connectedness, by which we mean that every two points of M can be joined by an arc[2] lying in M. And yet the convexity of M does not necessarily follow, but does that of the interior M^i of M, that is, we get the following result.

Theorem. *If M is locally convex and arcwise connected, then M^i is convex.*

2. Preliminaries.

The symbols $\{x, y\}$, $[x, y]$, etc. are defined as following.
Letting $z(\lambda)=(1-\lambda)x+\lambda y$ for $x \neq y$,

1) By $U(x\,;\,\varepsilon)$ we mean ε-neighborhood of x, that is,
$E[z \mid \|z-x\| < \varepsilon,\ z \in \Omega]$

2) The set C is called an *arc* if it is homeomorphic with the unit closed interval $[0, 1]$.

$$\{x, y\} = E\left[z(\lambda)\,\middle|\,\begin{matrix}\textit{for all real}\\ \textit{numbers } \lambda\end{matrix}\right], \qquad (x, y) = E[z(\lambda)\,|\,0<\lambda<1],$$
$$[x, y] = E[z(\lambda)\,|\,0\leq\lambda\leq 1], \qquad [x, y) = E[z(\lambda)\,|\,0\leq\lambda< 1],$$
$$(x, y] = E[z(\lambda)\,|\,0<\lambda\leq 1], \qquad (x, \overrightarrow{y}) = E[z(\lambda)\,|\,\lambda>1],$$
$$(\overleftarrow{x}, y) = E[z(\lambda)\,|\,\lambda<0],$$

where, of course,

$$\{x, y\} = \{y, x\}, \quad (x, y) = (y, x), \quad [x, y] = [y, x]$$
$$(x, \overrightarrow{y}) = (\overleftarrow{y}, x), \quad [x, y) = (y, x], \quad (x, y] = [y, x).$$

Let a and b be distinct points of Ω and let

$$c = (1-\alpha)a + \alpha b .$$

Lemma 1. *Given any* $\varepsilon > 0$, *we can find two positive numbers* $\delta_1 > 0$, $\delta_2 > 0$ *such that* $\{u, v\} \frown U(c; \varepsilon) \neq 0$ *for every* $u \in U(a; \delta_1)$ *and every* $v \in U(b; \delta_2)$. *Then it is said that* $U(a; \delta_1)$ *and* $U(b; \delta_2)$ *cross* $U(c; \varepsilon)$.

Proof. Set $\beta = Max.\left\{\left|\alpha - \frac{\varepsilon}{3\,\|a-b\|}\right|,\ \left|\alpha + \frac{\varepsilon}{3\,\|a-b\|}\right|\right\}$,

$$\gamma = Max.\left\{\left|1-\alpha+\frac{\varepsilon}{3\,\|a-b\|}\right|,\ \left|1-\alpha-\frac{\varepsilon}{3\,\|a-b\|}\right|\right\}.$$

For every u, v and λ such that $\|u-a\| < \delta_1 = \varepsilon/3\gamma$, $\|v-b\| < \delta_2 = \varepsilon/3\beta$, and $|\alpha-\lambda| < \varepsilon/3\,\|a-b\|$, it will be shown that $w = (1-\lambda)u + \lambda v$ belongs to $U(c; \varepsilon)$.

In fact, since $|\lambda| < \beta$ and $|1-\lambda| < \gamma$, we have

$$\|w-c\| \leq |1-\lambda|\ \|u-a\| + |\lambda|\ \|v-b\| + |\alpha-\lambda|\ \|a-b\|$$
$$< \gamma\cdot\frac{\varepsilon}{3\gamma} + \beta\cdot\frac{\varepsilon}{3\beta} + \frac{\varepsilon}{3\,\|a-b\|}\cdot\|a-b\| = \varepsilon .$$

Remark. If $0 < \varepsilon < 3\cdot Min.\{\|a-c\|, \|b-c\|\}$, then λ satisfies $(\lambda-1)(\alpha-1) > 0$ and $\lambda\alpha > 0$. We say that $U(a; \delta_1)$ *and* $U(b; \delta_2)$ *cross separately* $U(c; \varepsilon)$.

Corollary 1. *If for any* ε, $0 < \varepsilon < 2\cdot Min.\{\|a-c\|, \|b-c\|\}$, *we take any* v *and* λ *such that*

$$\|v-b\| < \delta_2 = \varepsilon/2\beta, \qquad |\alpha-\lambda| < \varepsilon/2\,\|a-b\|$$

where

$$\beta = Max.\left\{\left|\alpha-\frac{\varepsilon}{2\,\|a-b\|}\right|,\ \left|\alpha+\frac{\varepsilon}{2\,\|a-b\|}\right|\right\},$$

then $U(c\,;\,\varepsilon)$ *contains* $w=(1-\lambda)a+\lambda v$ *with* λ *satisfyiug both* $(\lambda-1)(\alpha-1)>0$ *and* $\lambda\alpha>0$.

It is said that a *and* $U(b\,;\,\delta_2)$ *cross separately* $U(c\,;\,\varepsilon)$.

Lemma 2. *Let* M *be a convex set,* $\overline{M}$ *the closure of* M. *If* $a\in M^i$ *and* $b\in\overline{M}$, *then* $[a,b)\subset M^i$ (Cf. [1]).

Proof. Suppose that $(a,b)\not\subset M^i$. Then (a,b) would contain $c\overline{\in} M^i$. Since $a\in M^i$, $U(a\,;\,\varepsilon)\subset M$ for some $\varepsilon<3\cdot Min.\{\|b-a\|,\ \|c-a\|\}$. By means of Lemma 1, we can find positive numbers ζ and η, such that $U(a\,;\,\varepsilon)$ is separately crossed by two neighborhoods: $U(b\,;\,\zeta)$ intersecting M, and $U(c\,;\,\eta)$ intersecting M'.[3] Letting $x\in U(b\,;\,\zeta)\frown M$, $z\in U(c\,;\,\eta)\frown M'$, and $y\in(x,\vec{z})\frown U(a\,;\,\varepsilon)\subset M$, we have $z\in[x,y]$, contrary to the convexity of M.

From this lemma we get at once:

Corollary 2. *Let* M *be a convex set,* M^* *its boundary, and,* M^e *its exterior. If* $a\in M^i$ *and* $r\in M^*$, *then* $(a,\vec{r})\subset M^e$.

Lemma 3. *If* $[a,b)\subset M^i$, *and* b *is a convex point of* M, *then there lies* $\delta>0$ *such that* $[a,z)\subset M^i$ *for any* $z\in U(b\,;\,\delta)\frown M$ (Cf. [2]).

Proof. To each $x\in[a,b]$, there corresponds $U(x\,;\,\varepsilon(x)/2)$ satisfying the following conditions:

$$U(x\,;\,\varepsilon(x))\subset M \quad \text{for} \quad x\in[a,b)\,,$$
$$U(x\,;\,\varepsilon(x))\frown M \ \text{ is convex for } \ x=b\,.$$

The system of $U(x\,;\,\varepsilon(x)/2)$ for all $x\in[a,b]$ covers $[a,b]$; however, since $[a,b]$ is compact, $[a,b]$ is covered by a finite system of $U_i=U(a_i\,;\,\varepsilon_i/2)$ $(i=1,\dots,n)$ where $\varepsilon_i=\varepsilon(a_i)$ and $a_i=(1-\alpha_i)a+\alpha_i b$ $(i=1,\dots,n)$ (Cf. [3]). Without loss of generality it may be assumed that

(i) $\alpha_1=0,\quad \alpha_n=1,\quad \alpha_i<\alpha_{i+1}\quad (i=1,\dots,n-1)$,
(ii) $U_i\not\subset U_j\quad (i\neq j)$, (iii) $U_i\subset M\quad (i=1,\dots,n-1)$,
(iv) $U_i\frown U_{i+1}\neq 0\quad (i=1,\dots,n-1)$.

As easily seen from them, we obtain

$$\frac{|\,\varepsilon_i-\varepsilon_j\,|}{2\,\|\,a-b\,\|}<|\,\alpha_i-\alpha_j\,| \quad \text{for} \quad i\neq j\,,$$

3) We denote by M' the complementary set of M

especially

$$\frac{|\varepsilon_i-\varepsilon_j|}{2\,\|a-b\|}<|\alpha_i-\alpha_j|<\frac{\varepsilon_i+\varepsilon_j}{2\,\|a-b\|} \quad \text{for} \quad i=j\pm 1\,,$$

and so the interval $[0, 1]$ is covered by the system of open sets:

$$V_i\equiv V_i(\alpha_i\,;\,\varepsilon_i/2\,\|a-b\|)=E[\lambda\,|\,|\lambda-\alpha_i|<\varepsilon_i/2\,\|a-b\|] \quad (i=1,\dots,n)\,.$$

Let

$$\delta=\underset{i=1,\dots,n}{Min.}\ \delta_i\,,$$

where

$$\delta_i=\varepsilon_i/2\beta_i\,,$$

$$\beta_i=Max.\left\{\left|\alpha_i-\frac{\varepsilon_i}{2\,\|a-b\|}\right|,\ \left|\alpha_i+\frac{\varepsilon_i}{2\,\|a-b\|}\right|\right\},\qquad (i=1,\dots,n)\,.$$

This δ will be what we desire here.

Setting $w(\lambda)\equiv(1-\lambda)a+\lambda z$ for any $z\in U(b\,;\,\delta)\frown M$, Corollary 1 shows that $w(\lambda)\in U(a_i\,;\,\varepsilon_i)$ for every $\lambda\in V_i$. Furthermore, if we take a real number ξ fulfilling

$$1-\frac{\varepsilon_n}{2\,\|a-b\|}<\xi<\alpha_{n-1}+\frac{\varepsilon_{n-1}}{2\,\|a-b\|}\,,$$

then for any $\lambda\in[0,\xi]$ there exists a positive integer k, i.e., one of $1, 2, \dots, n-1$ such that $\lambda\in V_k$, in other words, $w(\lambda)$ with any $\lambda\in[0,\xi]$ belongs to one of $U_i(i=1, 2, \dots, n-1)$; accordingly

$$[w(0)\,,\ w(\xi)]\subset M^i\,. \tag{1}$$

In particular, since $w(\xi)\in U_{n-1}\frown U_n\subset M$, $w(\xi)$ is an interior point of the convex set $U_n\frown$M.

Then by Lemma 2, it follows that

$$(w(\xi), z)\subset(U_n\frown M)^i\subset \mathrm{M}^i\,. \tag{2}$$

Combining with (1) and (2), we have $[a, z)\subset M^i$. Thus this lemma has been proved.

3. The proof of the theorem.

Let a and b be any distinct points of M^i. By the assumption a and b are joined by an arc C in M i.e.,

$$C=E[z\,|\,z=f(\lambda),\ 0\leq\lambda\leq 1]\subset M$$

where $f(\lambda)$ represents a homeomorphic image of λ in M.

Now, let us define $L(\lambda)$ as following:

$$L(0)=\{a\}, \qquad L(\lambda)=(a, f(\lambda)) \quad \text{for } \lambda \neq 0.$$

Evidently $L(0) \subset M^i$. Since a is an interior point of M and $f(\lambda)$ is continuous, for any $\varepsilon > 0$ there is β_0 such that $0 < \beta_0 < 1$, $f(\beta) \in U(a; \varepsilon) \subset M^i$ for all β, $0 < \beta < \beta_0$. Hence $L(\beta) \subset M^i$. Then we shall get $L(\lambda) \subset M^i$ for all $\lambda \in [0, 1]$, whence the proof of this theorem is to be finished.

Supposing that it is not true, there is one at least λ yielding $L(\lambda) \not\subset M^i$. First setting

$$\mu = \inf_{L(\lambda) \not\subset M^i} \lambda, \tag{4}$$

where μ clearly lies in $[\beta_0, 1]$, we shall prove that $L(\mu)$ i.e.,

$$L(\mu)=(a, f(\mu))=[x(\nu) \mid x(\nu)=(1-\nu)a+\nu f(\mu), \quad 0 < \nu < 1]$$

contains one at least point of M^*. To do this it is sufficient to show that only $L(\mu) \not\subset M^i$ because really $x(\nu) \in M^i$ for at least every $\nu \in [0, \varepsilon/\|a-f(\mu)\|]$. Suppose $L(\mu) \subset M^i$. Since $f(\mu)$ is a convex point of M, there exists $\delta > 0$ such that $(a, z) \subset M^i$ for any $z \in U(f(\mu); \delta)$ (by Lemma 3) and we take here particularly $z = f(\mu+\eta)$ such as shows below.
By continuity of $f(\lambda)$, we can select $\eta_0 > 0$ such that

$$f(\mu+\eta) \in U(f(\mu); \delta) \quad \text{for every } \eta \in (-\eta_0, \eta_0).$$

Therefore $L(\mu+\eta) \subset M^i$ for every $\eta \in (-\eta_0, \eta_0)$, contradicting to the assumption (4).

Let us denote by ν_0 the infinimum of all ν for which $x(\nu) \in M^* \cap L(\mu)$; obviously we have

$$\frac{\varepsilon}{\|a-f(\mu)\|} \leq \nu_0 < 1, \quad x(\nu_0) \in M^* \cap L(\mu)$$

and

$$x(\nu) \in M^i \quad \text{for every } \nu, \ 0 < \nu < \nu_0.$$

M is convex at $x(\nu_0)$, i.e., $U(x(\nu_0); \zeta) \cap M$ is convex for a suitable ζ; and if $\nu_0 - \zeta/\|a-f(\mu)\| < \nu < \nu_0$,

$$x(\nu) \in U(x(\nu_0); \zeta) \cap M^i = (U(x(\nu_0); \zeta) \cap M)^i.$$

On account of Corollary 2, it follows that

$$x(\xi) \in U(x(\nu_0); \zeta) \cap M^e \quad \text{for all } \xi, \ \nu_0 < \xi < \nu_0 + \zeta/\|a-f(\mu)\|,$$

that is,

$$U(x(\xi_0); \gamma) \subset U(x(\nu_0); \zeta) \cap M^e \quad \text{for some } \gamma > 0.$$

Hence a and $U(f(\mu);\sigma)$ cross $U(x(\xi_0);\gamma)$ if $\sigma>0$ is adequately chosen. On the other hand the continuity of $f(\mu)$ enables us to obtain $f(\mu-\delta)\in U(f(\mu);\sigma)$ for some $\delta>0$; so that a and $f(\mu-\delta)$ cross $U(x(\xi_0);\gamma)$, in other words, we have $L(\mu-\delta)\frown U(x(\xi_0);\gamma)\subset M^e$, i.e., $L(\mu-\delta)\not\subset M^i$, which arrives at the contradiction to $\mu=\inf_{L(\lambda)\not\subset M^i}\lambda$. Therefore $L(\lambda)\subset M^i$ for all $\lambda\in[0,1]$, especially, $L(1)=(a,b)\subset M^i$. The proof of the theorem has been completed.

We can easily give an example verifying that M is not convex under the same assumption as the above theorem. For example, let M be a set of points in the plane with cartesian coordinates $((x, y))$ satisfying

$$|y|\leqq 1 \quad \text{if} \quad \frac{1}{3}\leqq |x|\leqq 1\,,$$

$$|y|<1 \quad \text{if} \quad |x|<\frac{1}{3}\,.$$

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Notes.

〔1〕 By the way, it follows immediately from Lemma 2 that if M is convex M^i is convex. Moreover, it is likewise proved that if M is convex $\bar{M}$ is so.

〔2〕 Lemma 3 holds even if we let a be, more generally, a convex point of M.

〔3〕 Ω is regular and perfectly separable, because Ω is a separable metric space. Therefore Borel's covering theorem holds.