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ON LATTICOIDS.

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1. G. Birkhoff presented the following problem in his book[*]:

Problem 7. What are the consequences of weakening L 1 to

$$x \cup x = x \cap x$$

and L 4 to

$$x \cup (x \cap y) = x \cap (x \cup y)?$$

In this note we shall discuss the structure of the system on the results of weakening L 1 to

$$x \cup x = x \cap x$$

and L 4 to

$$x \cup (x \cap y) = x \cap (x \cup z).$$

Such a system is called a *latticoid* below.

Types of latticoids are seemed to be very complicated, and we cannot yet determine them, but in the special case (called simple) we shall give all types by means of the corresponding lattice (denoted as $\sigma(L)$) and a set of cardinal numbers.

Latticoids, above all, simple latticoids are seemed to be the most natural generalization of lattices, with respect to several aspects.

2. A set L of elements $a, b, c, \ldots$ which satisfies the following five conditions, is called a *latticoid*:

(0) Two binary operations $\cup$ and $\cap$ are defined to each ordered pairs a, b of L:

$$a, b \in L \quad \text{imply} \quad a \cup b \in L \text{ and } a \cap b \in L,$$

(1) $a \cup a = a \cap a$,

(2) $a \cup b = b \cup a$ and $a \cap b = b \cap a$,

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[*] G. Birkhoff: Lattice Theory p. 18, 1948.

(3) $(a \cup b) \cup c = a \cup (b \cup c)$, and $(a \cap b) \cap c = a \cap (b \cap c)$,

(4) $a \cup (a \cap b) = a \cap (a \cup c)$.

Note that the last condition (4) means two elements $a \cup (a \cap b)$ and $a \cap (a \cup b)$ are always equal, and do not depend upon b.

Put

$$\rho(a) = a \cup a = a \cap a,$$
$$\sigma(a) = a \cup (a \cap b) = a \cap (a \cup c).$$

Lemma 1.

(1) $\sigma(a) = a \asymp \rho(a) = a \cup a \cup a = a \cap a \cap a = a \cup (a \cap a) = a \cap (a \cup a)$.

(2) *Let $p(a)$ be a polynomial of a of degree n, which is greater than or equal to* 3, *then we have*

$$p(a) = \sigma(a),$$

(3) $a \asymp \sigma(a) = \sigma(a)$,

(4) $\sigma(a) \asymp \sigma(a) = \sigma(a)$,

(5) $\sigma(\sigma(a)) = \sigma(a)$,

(6) $\sigma(a \asymp b) = \sigma(a) \asymp \sigma(b) = \sigma(a) \asymp b = a \asymp \sigma(b)$.

(3), (4) *and* (5) *are the special cases of* (2).

Proof.

(1) By the definition of $\rho(a)$ and $\sigma(a)$,

(3) $a \cup \sigma(a) = a \cup (a \cap (a \cup a)) = \sigma(a)$ (the definition of $\sigma(a)$).

Dually we have $$a \cap \sigma(a) = \sigma(a),$$

(2) By the (n−4) iterations of (3).

(4) and (5) are only special cases of (2).

(6) $\sigma(a \cup b) = (a \cup b) \cup ((a \cup b) \cap b) = a \cup b \cup \sigma(b) = a \cup \sigma(b)$.

In the same way, we have

$$\sigma(a \cup b) = \sigma(a) \cup b,$$

and,

$$\begin{aligned} \sigma(a \cup b) &= \sigma(\sigma(a \cup b)) \quad \text{(By (5).)} \\ &= \sigma(a \cup \sigma(b)) \\ &= \sigma(a) \cup \sigma(b). \end{aligned}$$

Hence

$$\sigma(a \cup b) = \sigma(a) \cup \sigma(b) = \sigma(a) \cup b = a \cup \sigma(b).$$

Dually, we have

$$\sigma(a \cap b) = \sigma(a) \cap \sigma(b) = \sigma(a) \cap b = a \cap \sigma(b).$$

Theorem 1. *The sublatticoid $\sigma(L)$ of L is a lattice, where*

$$\sigma(L) = (\sigma(a)\,;\ a \in L).$$

Proof. $\sigma(L)$ is a subset of L, so $\sigma(L)$ satisfies the conditions (2) and (3) of the lattice. The preceding lemma shows that $\sigma(L)$ also satisfies the another conditions (0), (1) and (4) of the lattice.

Theorem 1′. *$\sigma(L)$ is the greatest lattice of all sublatticoids of L.*

Proof. If L' is a lattice contained in L, then $\sigma(L')=L'$. Now $L'\subseteq L$ implies $\sigma(L')\subseteq\sigma(L)$. Hence $L'=\sigma(L')\subseteq\sigma(L)$.

Theorem 1″. *A latticoid L is a lattice, if and only if*

$$L = \sigma(L).$$

Proof. If $L=\sigma(L)$, then L is a lattice (Theorem 1.). Conversely, if $L\neq\sigma(L)$, then L cannot be a lattice, for $\sigma(L)$ is the greatest lattice contained in L (Theorem 1′).

Theorem 2. *The mapping $\sigma : a\rightarrow\sigma(a)$ of L onto $\sigma(L)$ (or into L) is a lattice homomorphism in the sense that*

$$\sigma(a \asymp b) = \sigma(a) \asymp \sigma(b).$$

Proof. By the lemma 1, (6).

The mapping σ yields a partition of L, such that a and b belong to the same class, if and only if $\sigma(a)=\sigma(b)$.

In the partition by the mapping σ, we shall denote the class which contains $a \in L$ as $\bar{a}$ i. e.,

$$\bar{a} = (x\,;\ \sigma(x) = \sigma(a),\ x \in L).$$

We can easily see that

$$L \supseteq \rho(L) \supseteq \rho^2(L) = \sigma(L),$$

where

$$\rho^2(L) = \rho(\rho(L)).$$

Theorem 3. *Let a latticoid L be lattice homomorphic with a lattice L'. Then the lattice $\sigma(L)$ is lattice homomorphic with L'.*

Proof. Let f be the lattice homomorphic mapping of a latticoid L onto a lattice L'. Then we have

$$f(\sigma(a) \asymp \sigma(b)) = f(\sigma(a)) \asymp f(\sigma(b))\,,$$

for $\sigma(L)$ is a subset of L.

Hence f gives a lattice homomorphism of $\sigma(L)$ with L'.

Theorem 3′. *Let a latticoid L be lattice isomorphic with a lattice L′. Then L must be a lattice which is isomorphic with L′.*

Proof. $\sigma(L)$ is lattice homomorphic with L' by the Theorem 3. And this homomorphism must be a one-to-one mapping. This means that $\sigma(L)$ is lattice isomorphic with L'. But L is lattice isomorphic with L'. Then L is lattice isomorphic with $\sigma(L)$ by the mapping $a \rightarrow \sigma(a)$. Hence $L = \sigma(L)$. Therefore by the Theorem 1″ we can conclude that L is a lattice which is isomorphic with L'.

Theorem 4.

$$x \cup a = x \quad \textit{implies} \quad x = \sigma(x),$$

and dually

$$x \cap a = x \quad \textit{implies} \quad x = \sigma(x).$$

Proof. If

$$x \cup a = x,$$

then,

$$x = x \cup a = x \cup a \cup a = x \cup a \cup a \cup a = x \cup \sigma(a) = \sigma(x \cup a) = \sigma(x),$$

for

$$x \cup a = x.$$

In the same way, we have $x = \sigma(x)$, when $x \cap a = x$.

3. A latticoid L will be called *simple*, if for any two elements $a, b \in L$, always $a \asymp b \in \sigma(L)$.

Lemma 2. *If a latticoid L is simple, then*

$$a \asymp b = \sigma(a) \asymp b = a \asymp \sigma(b) = \sigma(a) \asymp \sigma(b) = \sigma(a \asymp b).$$

A latticoid L will be called *latticoid homomorphic* with a latticoid L', if there exists a mapping f of L onto L', such that

$$f(\sigma(a \asymp b)) = \sigma(f(a) \asymp f(b)).$$

In this case f is called a latticoid homomorphism of L with L'. If f is a one-to-one mapping, then the term homomorphism is replaced by isomorphism.

Theorem 5. *For any latticoid L, there exists a simple latticoid L′, with which L is latticoid isomorphic.*

Proof. A slight modification of definitions of $\cup$ and $\cap$ of L, such that

$$a \vee b = \sigma(a \cup b),$$

and

$$a \wedge b = \sigma(a \cap b)$$

yields a new latticoid L' with two operations $\vee$ and $\wedge$. It is easy to that L is latticoid isomorphic with L', and L' is a simple latticoid.

Theorem 6. *If a latticoid L is latticoid isomorphic with both simple latticoids L' and L'', then L' and L'' are lattice isomorphic with each other.*

Proof. If L is latticoid isomorphic with both L' and L'', by mappings f_1 and f_2, we have for each element pair $a, b \in L$,

$$\begin{aligned} f_j(a) \curlyvee\!\!\curlywedge f_j(b) &= \sigma(f_j(a) \curlyvee\!\!\curlywedge f_j(b)) && \text{(For, } L' \text{ and } L'' \text{ are both simple.)} \\ &= f_j(\sigma(a) \curlyvee\!\!\curlywedge \sigma(b)) \\ &= f_j(\sigma(a \curlyvee\!\!\curlywedge b)) && (j = 1, 2) \end{aligned}$$

Hence the mapping $g = f_2 f_1^{-1} : f_1(a) \to f_2(a)$, $a \in L$, gives a lattice isomorphism of L' with L'', that is, L' and L'' are lattice isomorphic with each other.

Remark. It can easily be led by the above theorem, that the two notions, lattice isomorphism and latticoid isomorphism, are coincides, so far as we shall concern with simple latticoids.

A *multiplicity* $\mathfrak{m}_a$ of an element a of L is the cardinal number of the class $\bar{a}$ of L.

Theorem 7. *If any lattice L, and a set of cardinal number $\mathfrak{m}_a$ corresponding to each element $a \in L$ are given, there exists a simple latticoid L', such that $\sigma(L')$ is lattice isomorphic with L and the multiplicity of each element $\sigma(a') \in \sigma(L')$ is $\mathfrak{m}_a$ corresponding to a, where a is a lattice isomorphic image of $\sigma(a')$.*

Proof. Take any element a of L, and construct a set $\bar{a}$, which contains a, and has cardinal number $\mathfrak{m}_a$.

Suppose $\bar{a}$ and $\bar{b}$ have no intersection, if $a \neq b$, and let L' be the set union of $\bar{a}$ for all a of L.

Then L' forms a simple latticoid with operations

$$x \curlyvee\!\!\curlywedge y = a \curlyvee\!\!\curlywedge b,$$

where

$$x \in \bar{a}, \quad y \in b,$$

and

$$L' \supseteq \sigma(L') = L.$$

It is obvious that the multiplicity of each element $x \in \bar{a}$, is $\mathfrak{m}_a$.

Theorem 8. *A simple latticoid L is determined up to lattice isomorphism by means of the lattice $\sigma(L)$ and a set of multiplicity of each element of $\sigma(L)$.*

Proof. Let L_1 and L_2 be two simple latticoids, and $\sigma(L_1)$ and $\sigma(L_2)$ be lattice isomorphic with each other. Moreover let the multiplicity of each element of $\sigma(L_1)$ and that of corresponding element of $\sigma(L_2)$ be the same. The assumption of this theorem enables us to extend the lattice isomorphic mapping between $\sigma(L_1)$ and $\sigma(L_2)$ to a lattice isomorphic mapping between whole L_1 and L_2, naturally:

if $a_1 \leftrightarrow a_2$, $a_1 \in \sigma(L_1)$, $a_2 \in \sigma(L_2)$,

then the cardinal number of $\bar{a}_1$ *and* $\bar{a}_2$ *are the same.*

Thus we can construct a one-to-one mapping between $\bar{a}_1$ *and* $\bar{a}_2$ *so a: to* a_1 *correspond to* a_2.

This extended mapping between L_1 and L_2 must be a lattice isomorphism between them:

if $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$, *then* $\sigma(x_1) \leftrightarrow \sigma(x_2)$, $\sigma(y_1) \leftrightarrow \sigma(y_2)$.

Therefore $x_1 \curlywedge y_1 = \sigma(x_1) \curlywedge \sigma(y_1) \leftrightarrow \sigma(x_2) \curlywedge \sigma(y_2) = x_2 \curlywedge y_2$.

A slight modification of the proof of the preceding theorem leads the following.

Theorem 9. *A latticoid L is completely determined up to latticoid isomorphism by means of the lattice* $\sigma(L)$ *and a set of multiplicity of each element of* $\sigma(L)$.

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