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ON THE BEHAVIOUR OF POWER SERIES ON THE BOUNDARY OF THE SPHERE OF ANALYTICITY IN ABSTRACT SPACES

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In classical analysis there exists a singular point at least on the circle of convergence of the power series, but this is not true generally in the case of the power series in complex-Banach-spaces. In this paper we shall investigate a necessary and sufficient condition for power series in complex-Banach-spaces to be analytic at all points on the boundary of the sphere of analyticity.

Let E and E' be two complex-Banach-spaces and an E'-valued function $h_n(x)$ defined on E be a homogeneous polynomial of degree n. Then the radius of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$ exsists, which is written by τ*). The sphere $\|x\|<\tau$ is called the sphere of analyticity of the power series $\sum_{n=0}^{\infty} h_n(x)$.

Theorem 1. *In order that $\sum_{n=0}^{\infty} h_n(x)$ is analytic at all points on the boundary of the sphere of analyticity, it is necessary and sufficient that*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} < \frac{1}{\tau} \tag{1}$$

for an arbitrary compact set G extracted from the set $\|x\|=1$.

Proof. Let $\sum_{n=0}^{\infty} h_n(x)$ be analytic at all points on $\|x\|=\tau$. If a compact set G extracted from $\|x\|=1$ exists which stisfies the following equality

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau},$$

we have

$$\frac{1}{\tau+\varepsilon_i} < \sqrt[n_i]{\sup_{x\in G} \|h_{n_i}(x)\|} \tag{2}$$

*) Isae Shimoda, On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.

for a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_n > \dots$, which tends to zero, and n_i which corresponds to ε_i, where $i=1, 2, \dots, n, \dots$. Since G is compact, there exists x_i in G which satisfies

$$\sup_{x \in G} \|h_{n_i}(x)\| = \|h_{n_i}(x_i)\|.$$

Then we can select from $\{x_i\}$ a subsequence which converges to x_0, of course $x_0 \in G$. In order not to change notation we shall suppose simply that the sequence $\{x_i\}$ itself converges to x_0.

Put $(\tau+\varepsilon_i)x_i = y_i$ and $\tau x_0 = y_0$, then y_i converges to y_0. From (2), we have

$$1 < \|h_{n_i}(y_i)\|. \tag{3}$$

where $i=1, 2, 3, \dots$.

Let M be a compact set composed of $y_0 e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Since $\sum_{n=0}^{\infty} h_n(x)$ is analytic on M, we can find a finite system of neighbourhoods U_j of $y_0 e^{i\theta_j}$ $(j=1, 2, \dots, n_0)$ such that $\sum_{j=1}^{n_0} U_j$ covers M and $\|\sum_{n=0}^{\infty} h_n(y)\| \leq N$ for $y \in \sum_{j=1}^{n_0} U_j$. Now we choose two small positive numbers δ and ρ, so that $y\alpha \in \sum_{j=1}^{n_0} U_j$, where $\|y-y_0\| \leq \rho$ and $|\alpha| = 1+\delta$. Then we have

$$\|h_n(y)\| = \left\| \frac{1}{2\pi i} \int_{|\alpha|=1+\delta} \frac{\sum_{n=0}^{\infty} h_n(\alpha y)}{\alpha^{n+1}} d\alpha \right\| \leq \frac{N}{(1+\delta)^n} \tag{4}$$

for $n=1, 2, \dots$ and $\|y-y_0\| < \rho$.
Since y_i converges to y_0, (4) contradicts to (3). This shows that the condition (1) is necessary.

Let y_0 be an arbitrary point on $\|y\| = \tau$. Suppose that there exists a sequence $\{y_n\}$ which converges to y_0 and satisfies the following inequalities

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(y_i)\|} \geq 1 - \varepsilon_i \tag{5}$$

for $i=1, 2, \dots$, where a sequence of positive numbers $\{\varepsilon_n\}$ converges to zero with $\varepsilon_{n+1} < \varepsilon_n$. Put $\frac{y_i}{\|y_i\|} = x_i$ and $\{x_i\} = G$. Then G is a compact set on $\|x\| = 1$. Now we assume (1). Then there exists a positive number ε such that $\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} \leq \frac{1}{\tau+3\varepsilon}$. From this, we have $\|h_n(x_i)\| \leq \frac{1}{(\tau+2\varepsilon)^n}$, for $n \geq n_0$ and $i=1, 2, \dots$. Since $x_i = \frac{y_i}{\|y_i\|}$, $\|h_n(y_i)\| \leq \left(\frac{\|y_i\|}{\tau+2\varepsilon}\right)^n$, for $n \geq n_0$ and $i=1, 2, \dots$. On the other hand, there exists N such that $\|y_i\| < \tau+\varepsilon$ for $i \geq N$, because $y_i \to y_0$ and $\|y_0\| = \tau$. Thus we have

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(y_i)\|}\leq\frac{\|y_i\|}{\tau+2\varepsilon}\leq\frac{\tau+\varepsilon}{\tau+2\varepsilon}<1$$

for $i\geq N$, contradicting to (5). From this we can easily see that there exist two positive numbers δ and ε such that $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(y)\|}\leq 1-\varepsilon$ uniformly for $\|y-y_0\|<\delta$. Hence, $\sum_{n=0}^{\infty} h_n(x)$ is uniformly convergent in $\|y-y_0\|<\delta$ and then $\sum_{n=0}^{\infty} h_n(x)$ is analytic in $\|y-y_i\|<\delta$. This completes the proof.

An example is afforded which is analytic at all points on the boundary of the sphere of analyticity. Put $h_n(x)=\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n$, where $x=(x_1, x_2, \dots)$ is a point of complex-l_2-spaces, and $h_n(x)$ takes complex numbers as its values. Then the radius of analyticity of $\sum_{n=0}^{\infty} h_n(x)$ is 1 and yet $\sum_{n=2}^{\infty} h_n(x)$ is analytic everywhere on $\|x\|=1$.

The radius of analyticty of $\sum_{n=2}^{\infty} h_n(x)$ is given by

$$\frac{1}{\tau}=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\ ^{*)}$$

Since $\|x\|=1$, $|x_i|\leq 1$ for $i=1, 2, \dots$. Therefore

$$\left.\begin{aligned}\frac{1}{\tau}&=\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\left|\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n\right|}\\&\leq\overline{\lim_{n\to\infty}}\sqrt[n]{n\left(1-\frac{1}{n}\right)^n}\\&=1\end{aligned}\right\}\qquad(6)$$

Now put $X_m=(0,\dots, 0, 1, 0,\dots)$, where only m-th coordinate is 1 and others are all zero. Since $\|X_m\|=1$, we have

$$\frac{1}{\tau}\geq\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(X_m)\|}=\overline{\lim_{n\to\infty}}\sqrt[n]{\left(1-\frac{1}{m}\right)^n}=1-\frac{1}{m},$$

for $m=2, 3, \dots$.

Hence from (6), we see that $\tau=1$.

Let G be an arbitrary compact set on $\|x\|=1$, then there exists the convergent series of non negative constant $\sum_{n=1}^{\infty} a_n^2=1$ such that $\sum_{n=m}^{\infty}|x_n|^2<\sum_{n=m}^{\infty} a_n^2$ for $x\in G$ and $m=1, 2, 3, \dots$. If $a_1=a_2=a_3=\cdots=a_{n_0}=0$ and $a_{n_0+1}\neq 0$, $|x_m|^2\leq 1$ for $m=1, 2, \dots, n_0+1$ and $|x_m|^2<\sum_{n=m}^{\infty}|x_n|^2<\sum_{n=n_0+2}^{\infty} a_n^2<1$ for

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$m \geq n_0+2$. Put $\delta = \max\left(1-\frac{1}{n_0+1},\ \sqrt{\sum_{n_0+2}^{\infty} a_n^2}\right)$, then $\delta < 1$. Thus we have $\|h_n(x)\| = |\sum_{m=2}^{n}\left(1-\frac{1}{m}\right)^n x_m^n| < \sum_{m=2}^{n_0+1}\left(1-\frac{1}{m}\right)^n + \sum_{m=n_0+2}^{n} |x_m|^n < n\delta^n$. Hence, $\overline{\lim}_{n\to\infty} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} \leq \delta < 1$. Thus Theorem 1 is applicable, and we see that $\sum_{n=2}^{\infty} h_n(x)$ is analytic everywhere on the boundary of the sphere of analyticity.

Theorem 2. *In order that there exists at least a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on the boundary of the sphere of analyticity, it is necessary and sufficient to exist at least a compact set G on $\|x\|=1$ which satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau} \tag{7}$$

Proof. If there does not exist a singular point of $\sum_{n=0}^{\infty} h_n(x)$ on $\|x\|=\tau$, it must be analytic on $\|x\|=\tau$. By appealing to Theorem 1, we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} < \frac{1}{\tau},$$

in contradiction to our assumption that $\overline{\lim}_{n\to\infty} \sqrt[n]{\sup_{x\in G} \|h_n(x)\|} = \frac{1}{\tau}$. The inverse is proved as well.

Similarly we have Theorem 3 from Theorem 1.

Theorem 3. *If a power series $\sum_{n=0}^{\infty} h_n(x)$ is analytic at all points on the boundary of its sphere of analyticity $\|x\|=\tau$, then we have*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} < \frac{1}{\tau} \tag{8}$$

for an arbitrary point x on $\|x\|=1$.

From Theorem 3, we have following theorem.

Theorem 4. *If a point x, which lies on $\|x\|=1$, satisfies the following equality*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} = \frac{1}{\tau},$$

then there exists at least a singular point on $\|x\|=\tau$.

The condition (8) is necessary for $\sum_{n=0}^{\infty} h_n(x)$ to be analytic on the boundary of its sphere of analyticity, but is not sufficient as the following example shows.

Put $h_n(X)=x^{n-1}y$ in the complex-2-dimensional spaces, then $h_n(X)$ is a homogeneous polynomial of degree n, where $X=(x, y)$. We have

$$\begin{aligned}\sup_{\|X\|=1} \sqrt[n]{\|h_n(X)\|} &= \sup_{\|X\|=1} \overline{\lim_{n\to\infty}} |x|^{\frac{n-1}{n}} |y|^{\frac{1}{n}} \\ &= \sup_{\|X\|=1} |x| \\ &= 1\end{aligned}$$

That is, the radius of analyticity of $\sum_{n=1}^{\infty} h_n(X)$ is 1. Now let G be a compact set on $\|X\|=1$ composed of $X_0=(1,0)$ and $X_m=\left(\sqrt{1-\frac{1}{m}},\ \sqrt{\frac{1}{m}}\right)$, with $m=1, 2, \ldots$.

Then we have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{X\in G} \|h_n(X)\|} = \overline{\lim_{n\to\infty}} \sqrt[2n]{\left(1-\frac{1}{n}\right)^{n-1} \frac{1}{n}} = 1,$$

because $(1-t)^{n-1}t$ takes its maximum at $t=\frac{1}{n}$ in the interval $0\leq t\leq 1$. Thus Theorem 2 is applicable and we see that $\sum_{n=1}^{\infty} h_n(X)$ has a singular point on the boundary of its sphere of analyticity.

On the other hand, we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(X)\|} &= \overline{\lim_{n\to\infty}} |x|^{\frac{n-1}{n}} |y|^{\frac{1}{n}} \\ &= |x|<1, \text{ for } y\neq 0 \text{ on } \|X\|=1, \\ &= 0<1, \quad \text{for } y=0 \text{ on } \|X\|=1.\end{aligned}$$

This shows that (8) is satisfied.

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